

# Geometrical Tools for Quantum Euclidean Spaces

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## Abstract

We apply one of the formalisms of noncommutative geometry to  $\mathbb{R}_q^N$ , the quantum space covariant under the quantum group  $SO_q(N)$ . Over  $\mathbb{R}_q^N$  there are two  $SO_q(N)$ -covariant differential calculi. For each we find a frame, a metric and two torsion-free covariant derivatives which are metric compatible up to a conformal factor and which have a vanishing linear curvature. This generalizes results found in a previous article for the case of  $\mathbb{R}_q^3$ . As in the case  $N = 3$ , one has to slightly enlarge the algebra  $\mathbb{R}_q^N$ ; for  $N$  odd one needs only one new generator whereas for  $N$  even one needs two. As in the particular case  $N = 3$  there is a conformal ambiguity in the natural metrics on the differential calculi over  $\mathbb{R}_q^N$ . While in our previous article the frame was found ‘by hand’, here we disclose the crucial role of the quantum group covariance and exploit it in the construction. As an intermediate step, we find a homomorphism from the cross product of  $\mathbb{R}_q^N$  with  $U_q so(N)$  into  $\mathbb{R}_q^N$ , an interesting result in itself.

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# 1 Introduction

It is an old idea [33, 34] that a noncommutative modification of the algebraic structure of space-time could provide a regularization of the divergences of quantum field theory, because the representations of noncommutative ‘spaces’ have a lattice-like structure [25, 22, 26, 20, 5] which should automatically impose an ultraviolet cut-off [14, 9]. This idea has been challenged recently from various points of view [21, 8, 28, 29, 32]. In any case to discuss it and other problems it is necessary to have a noncommutative version of flat space [19, 31, 27, 4, 7]. Two approaches have been suggested to endow an algebra with the noncommutative generalization of a differential calculus, that of Woronowicz [37, 38, 39] and that of Connes [11, 12]. The formalism which we shall use here is an attempt to conciliate these two points of view in a particular class of examples. To this end we use a particular noncommutative version [13] of the moving-frame formalism of E. Cartan.

In a previous article a detailed study was made of the noncommutative geometry of  $\mathbb{R}_q^3$ , the quantum space covariant under the quantum group  $SO_q(3)$ . It was found that one had to slightly extend the algebra by adding a ‘dilatator’  $\Lambda$  in order to reduce the center to the complex numbers and so to be able to construct an essentially unique metric, whereas for the construction of a frame one also had to add the square roots and the inverses of some generators.

The results are here extended to the case of the general algebra  $\mathbb{R}_q^N$ . We find that the cases  $N$  even and  $N$  odd are somewhat different. When  $N$  is odd the formalism is quite similar to the case  $N = 3$ . When, on the other hand,  $N$  is even, yet another extension must be made for the construction of a frame. We must add to the algebra one of the components  $K$  of the angular momentum in order to have a trivial center. The differential calculus of [3] is extended by setting  $dK = 0$  and either  $d\Lambda = 0$  or  $d\Lambda \neq 0$  but fixed by a modified Leibniz rule. These extensions are in a sense unsatisfactory since they imply that there are elements of the extended algebra (what we shall call  $\mathcal{A}_N$ ) which have vanishing derivative but which are nonetheless noncommutative analogues of non-constant functions. Their inclusion can be interpreted as an embedding of the ‘configuration space’ into part of ‘phase space’. The different possibilities lead to a conformal ambiguity in the natural metrics on the differential calculi over  $\mathbb{R}_q^N$ ; one choice favours the geometry  $S^{N-1} \times \mathbb{R}$  and the second, the one we emphasize here, favours the flat geometry  $\mathbb{R}^N$ . For each of its two  $SO_q(N)$ -covariant differential calculi we find the corresponding frame and two torsion-free covariant derivatives that are metric compatible up to a conformal factor and which yield both a vanishing linear curvature. Apart from a few notes, we leave the study of the reality structures,  $*$ -representations and of the commutative limit as subjects to be treated elsewhere.

In Section 2 we briefly recall the tools of noncommutative geometry [12]

which will be needed. We start with a formal noncommutative algebra  $\mathcal{A}$  and with a differential calculus  $\Omega^*(\mathcal{A})$  over it. We define then a frame or ‘Stehbein’ [13] and the corresponding metric and covariant derivative. We also recall how a generalized Dirac operator [12] can be constructed from the frame and a dual set of inner derivations. Finally we recall the compatibility condition between the metric and the covariant derivative. The frame is what will permit us to pass from the covariant definition of a differential calculus [36], with its emphasis on  $q$ -deformed commutators, to the definition of Connes, which uses ordinary commutators, at least on a formal level; we do not attempt to discuss the Dirac operator as an operator on a graded Hilbert space.

In Section 3 we recall some formulae from the pioneering work of Faddeev, Reshetikhin and Takhtajan [19] on the definition of  $\mathbb{R}_q^N$  as given by the coaction of the quantum group  $SO_q(N)$ . We give then a brief overview of the work of Carow-Watamura, Schlieker, Watamura [3] and Ogievetski [30] on the construction of two differential calculi on  $\mathbb{R}_q^N$ , which are based on the  $\hat{R}$ -matrix formalism and are covariant with respect to  $SO_q(N)$ . They both yield the de Rham calculus in the commutative limit. However here it is convenient to formulate quantum group covariance in terms of the action on  $\mathbb{R}_q^N$  of the dual Hopf algebra  $U_q so(N)$ , rather than in terms of the coaction of  $SO_q(N)$ .

In Section 4 we proceed with the actual construction of the frame over  $\mathbb{R}_q^N$  and of the inner derivations dual to it. We first solve the problem in a larger algebra,  $\Omega^*(\mathcal{A}_N) \rtimes U_q so(N)$ , where we show that a frame has to transform under the action of the quantum group  $U_q^{op} so(N)$  with opposite coalgebra. We also find a dual set of inner derivations by decomposing in the frame basis the formal ‘Dirac operator’, which had already been found [10, 35] previously. It would be interesting, but requires some non-trivial handwork, to compare our results with the ones of Ref. [1, 2]. There multiparametric deformations of the inhomogeneous  $SO_q(N)$  quantum groups are considered, whereby multiparametric deformations (including as a particular case the one-parameter one at hand) of the Euclidean space are obtained by projection. The frame of the quantum group in the Woronowicz bicovariant differential calculi sense, i.e. the left- (or right-) invariant 1-forms, might also be projected and compared to ours.

Then in Section 5 we show that it is possible to find homomorphisms  $\varphi^\pm : \mathcal{A}_N \rtimes U_q^\pm so(N) \rightarrow \mathcal{A}_N$  which act trivially on the factor  $\mathcal{A}_N$  on the left-hand side and which project the components of the frame and of the inner derivations from elements of  $\mathcal{A}_N \rtimes U_q so(N)$  onto elements in  $\mathcal{A}_N$ . This implies that in the  $x^i$  basis they satisfy the ‘RLL’ and the ‘gLL’ relations fulfilled by the  $L^\pm$  [19] generators of  $U_q so(N)$ . In the case that  $N$  is odd it is possible to ‘glue’ the homomorphisms together to an isomorphism from the whole of  $\mathcal{A}_N \rtimes U_q so(N)$  to  $\mathcal{A}_N$ , an interesting and surprising result in itself.

Finally in Section 6 we see that for each of the two calculi there is

essentially a unique metric, and two torsion-free  $SO_q(N)$ -covariant linear connections which are compatible with it up to a conformal factor.

## 2 The Cartan formalism

In this section we briefly review a noncommutative extension [13] of the moving-frame formalism of E. Cartan. We start with a formal noncommutative associative algebra  $\mathcal{A}$  with a differential calculus  $\Omega^*(\mathcal{A})$ . If  $\mathcal{A}$  has a commutative limit and if this limit is the algebra of functions on a manifold  $M$  then we suppose that the limit of the differential calculus is the ordinary  $\Omega^*(\mathcal{A}_N) \rtimes U_q so(N)$  de Rham differential calculus on  $M$ . We shall concentrate on the case where the module of the 1-forms  $\Omega^1(\mathcal{A})$  is free of rank  $N$  as a left or right module and admits a special basis  $\{\theta^a\}_{1 \leq a \leq N}$ , referred to as ‘frame’ or ‘Stehbein’, which commutes with the elements of  $\mathcal{A}$ :

$$[f, \theta^a] = 0. \quad (2.1)$$

This means that if the limit manifold exists it must be parallelizable. The integer  $N$  plays the role of the dimension of the manifold. We suppose further that the basis  $\theta^a$  is dual to a set of inner derivations  $e_a = \text{ad } \lambda_a$  such that:

$$df = e_a f \theta^a = [\lambda_a, f] \theta^a \quad (2.2)$$

for any  $f \in \mathcal{A}$ . The formal ‘Dirac operator’ [12], defined by the equation

$$df = -[\theta, f], \quad (2.3)$$

is then given by

$$\theta = -\lambda_a \theta^a. \quad (2.4)$$

We shall consider only the case where the center  $\mathcal{Z}(\mathcal{A})$  of  $\mathcal{A}$  is trivial:  $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$ . If the original algebra does not have a trivial center then we shall extend it an algebra which does. The (wedge) product  $\pi$  in  $\Omega^*(\mathcal{A})$  can be defined by relations of the form

$$\theta^a \theta^b = P^{ab}_{cd} \theta^c \otimes \theta^d \quad (2.5)$$

for suitable  $P^{ab}_{cd} \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}$ . It can be shown that consistency with the nilpotency of  $d$  requires that the  $\lambda_a$  satisfy a quadratic relation of the form

$$2\lambda_c \lambda_d P^{cd}_{ab} - \lambda_c F^c_{ab} - K_{ab} = 0. \quad (2.6)$$

The coefficients of the linear and constant terms must also belong to the center. In the cases which interest us here they vanish. Notice that Equation (2.6) has the form of the structure equation of a Lie algebra with a central extension.

We define [17] the metric as a non-degenerate  $\mathcal{A}$ -bilinear map

$$g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \mathcal{A}. \quad (2.7)$$

This means that it can be completely determined up to central elements once its action on a basis of 1-forms is assigned. For example set

$$g(\theta^a \otimes \theta^b) = g^{ab}. \quad (2.8)$$

The bilinearity implies that

$$fg^{ab} = g(f\theta^a \otimes \theta^b) = g(\theta^a \otimes \theta^b f) = g^{ab}f$$

and therefore  $g^{ab} \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}$ :  $\{\theta^a\}$  is a special basis of 1-forms in which the coefficients of the metric are central elements, namely complex numbers in our assumptions. This is the property characterizing frames (vielbein) in ordinary geometry, and is at the origin of the name ‘frame’ for this basis also in noncommutative geometry. To define a covariant derivative  $D$  which satisfies [17] a left and right Leibniz rule we introduce a ‘generalized flip’, an  $\mathcal{A}$ -bilinear map

$$\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}). \quad (2.9)$$

The flip is also completely determined once its action on a basis of 1-forms is assigned. For example set

$$\sigma(\theta^a \otimes \theta^b) = S^{ab}_{cd} \theta^c \otimes \theta^d. \quad (2.10)$$

As above, bilinearity implies that  $S^{ab}_{cd} \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}$ . Using the flip a left and right Leibniz rule can be written:

$$D(f\xi) = df \otimes \xi + fD\xi \quad (2.11)$$

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f. \quad (2.12)$$

The torsion map

$$\Theta : \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A}) \quad (2.13)$$

is defined by

$$\Theta = d - \pi \circ D. \quad (2.14)$$

We shall assume that  $\sigma$  satisfies the condition

$$\pi \circ (\sigma + 1) = 0 \quad (2.15)$$

in order that the torsion be bilinear. The usual torsion 2-form  $\Theta^a$  is defined as  $\Theta^a = d\theta^a - \pi \circ D\theta^a$ . It is easy to check [17] that if on the right-hand side of Equation (2.6) the term linear in  $\lambda_a$  and the constant term vanish then a torsion-free covariant derivative can be defined by

$$D\xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta), \quad (2.16)$$

for any  $\xi \in \Omega^1(\mathcal{A})$ . The most general torsion-free  $D$  for fixed  $\sigma$  is of the form

$$D = D_{(0)} + \chi \quad (2.17)$$

where  $\chi$  is an arbitrary  $\mathcal{A}$ -bimodule morphism

$$\Omega^1(\mathcal{A}) \xrightarrow{\chi} \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \quad (2.18)$$

fulfilling

$$\pi \circ \chi = 0. \quad (2.19)$$

The compatibility of a covariant derivative with the metric is expressed by the condition [18]

$$g_{23} \circ D_2 = d \circ g. \quad (2.20)$$

For the covariant derivative (2.16) this condition can be written as the equation

$$S^{ae}_{df} g^{fg} S^{bc}_{eg} = g^{ab} \delta_d^c \quad (2.21)$$

if one uses the coefficients of the flip with respect to the frame.

Introduce the standard notation  $\sigma_{12} = \sigma \otimes \text{id}$ ,  $\sigma_{23} = \text{id} \otimes \sigma$ , to extend to three factors of a module any operator  $\sigma$  defined on a tensor product of two factors. There is a natural continuation of the map (2.9) to the tensor product  $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$  given by the map

$$D_2(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta). \quad (2.22)$$

We define formally the curvature as

$$\text{Curv} \equiv D^2 = \pi_{12} \circ D_2 \circ D. \quad (2.23)$$

We recover the standard definition of the frame components  $R^a_{bcd}$  of the curvature tensor from the decomposition

$$\text{Curv}(\theta^a) = -\frac{1}{2} R^a_{bcd} \theta^c \theta^d \otimes \theta^b \quad (2.24)$$

One can easily show [23] that the curvature associated to (2.16) is given by

$$\text{Curv}(\xi) = \xi_a \theta^2 \otimes \theta^a + \pi_{12} \sigma_{12} \sigma_{23} \sigma_{12} (\xi \otimes \theta \otimes \theta). \quad (2.25)$$

The algebra we shall consider is a  $*$ -algebra. We shall require that the involution  $*$  be extendable to the algebra of differential forms in such a way that

$$(\xi \eta)^* = (-1)^{pq} \eta^* \xi^*, \quad \xi \in \Omega^p(\mathcal{A}), \quad \eta \in \Omega^q(\mathcal{A}). \quad (2.26)$$

We recall that the elements of the algebra are considered as 0-forms. One would like to have a differential fulfilling the reality condition

$$(df)^* = df^* \quad (2.27)$$

as in the commutative case. Neither of the two differential calculi we shall introduce in Section 3 satisfies this condition; the differential calculus  $\Omega^*(\mathcal{A})$  is mapped by  $*$  into a new one  $\bar{\Omega}^*(\mathcal{A})$ . As a consequence, the reality conditions on the covariant derivative and curvature formulated in [24] cannot be satisfied. However we shall still suppose [18] that the extension of the involution to the tensor product is given by

$$(\xi \otimes \eta)^* = \sigma(\eta^* \otimes \xi^*). \quad (2.28)$$

A change in  $\sigma$  therefore implies a change in the definition of an hermitian tensor. The reality condition for the metric will be, as in [24],

$$g \circ \sigma(\eta^* \otimes \xi^*) = (g(\xi \otimes \eta))^*. \quad (2.29)$$

We shall also continue to assume that  $\sigma$  satisfies the braid equation

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}, \quad (2.30)$$

a condition implied [24] by the reality condition on the covariant derivative and the curvature. At the end of Section 6 we shall briefly consider the question how to modify reality condition on the covariant derivative and the curvature in the present case.

### 3 The quantum Euclidean spaces and their $q$ -deformed differential calculi

The starting point for the definition of the  $N$ -dimensional quantum Euclidean space  $\mathbb{R}_q^N$  is the braid matrix  $\hat{R}$  for  $SO_q(N, \mathbb{C})$  a  $N^2 \times N^2$  matrix, whose explicit expression we give in Appendix 7.1. Certain properties of  $\hat{R}$  which we shall use follow immediately from the definition. First, it fulfills the braid equation

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}. \quad (3.1)$$

Here we have used again the conventional tensor notation  $\hat{R}_{12} = \hat{R} \otimes \text{id}$ ,  $\hat{R}_{23} = \text{id} \otimes \hat{R}$ . By repeated application of the Equation (3.1) one finds

$$f(\hat{R}_{12}) \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} f(\hat{R}_{23}) \quad (3.2)$$

for any polynomial function  $f(t)$  in one variable. The Equations (3.1) and (3.2) are evidently satisfied also after the replacement  $\hat{R} \rightarrow \hat{R}^{-1}$ . Second,  $\hat{R}$  is invariant under transposition of the indices:

$$\hat{R}_{kl}^{ij} = \hat{R}_{ij}^{kl}. \quad (3.3)$$

Here and in the sequel we use indices with values

$$\begin{aligned} i &= -n, \dots, -1, 0, 1, \dots, n, \quad \text{with } n \equiv \frac{N-1}{2} \text{ for } N \text{ odd,} \\ i &= -n, \dots, -1, 1, \dots, n, \quad \text{with } n \equiv \frac{N}{2} \text{ for } N \text{ even.} \end{aligned} \quad (3.4)$$

and  $n$  to denote the rank of  $SO(N, \mathbb{C})$ . The matrix element  $\hat{R}_{kl}^{ij}$  vanishes unless the indices satisfy the following condition:

$$\begin{aligned} & \text{either } i \neq -j \text{ and } k = i, l = j \text{ or } l = i, k = j \\ & \text{or } i = -j \text{ and } k = -l. \end{aligned} \quad (3.5)$$

The  $R$ -matrix, defined by

$$R_{kl}^{ij} = \hat{R}_{kl}^{ji},$$

is lower-triangular.

There exists also [19] a projector decomposition of  $\hat{R}$ :

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{1-N}\mathcal{P}_t. \quad (3.6)$$

The  $\mathcal{P}_s$ ,  $\mathcal{P}_a$ ,  $\mathcal{P}_t$  are  $SO_q(N)$ -covariant  $q$ -deformations of the symmetric trace-free, antisymmetric and trace projectors respectively and satisfy the equations

$$\mathcal{P}_\mu \mathcal{P}_\nu = \mathcal{P}_\mu \delta_{\mu\nu}, \quad \sum_\mu \mathcal{P}_\mu = 1, \quad \mu, \nu = s, a, t. \quad (3.7)$$

The  $\mathcal{P}_t$  projects on a one-dimensional sub-space and therefore it can be written in the form

$$\mathcal{P}_{t\,kl}^{ij} = (g^{sm} g_{sm})^{-1} g^{ij} g_{kl} = \frac{k}{\omega_n(q^{-\rho_n+1} - q^{\rho_n-1})} g^{ij} g_{kl} \quad (3.8)$$

where  $g_{ij}$  is the  $N \times N$  matrix

$$g_{ij} = q^{-\rho_i} \delta_{i,-j}. \quad (3.9)$$

We have here introduced the notation

$$\rho_i = \begin{cases} (n - \frac{1}{2}, \dots, \frac{1}{2}, 0, -\frac{1}{2}, \dots, \frac{1}{2} - n) & \text{for } N \text{ odd,} \\ (n - 1, \dots, 0, 0, \dots, 1 - n) & \text{for } N \text{ even.} \end{cases}$$

and we have set

$$k \equiv q - q^{-1}, \quad \omega_i \equiv q^{\rho_i} + q^{-\rho_i}.$$

The matrix  $g_{ij}$  is a  $SO_q(N)$ -isotropic tensor and is a deformation of the ordinary Euclidean metric in a set of coordinates pairwise conjugated to each other under complex conjugation. It is easily verified that its inverse  $g^{ij}$  is given by

$$g^{ij} = g_{ij}. \quad (3.10)$$

The metric and the braid matrix satisfy the relations [19]

$$g_{il} \hat{R}^{\pm 1lh}_{jk} = \hat{R}^{\mp 1hl}_{ij} g_{lk}, \quad g^{il} \hat{R}^{\pm 1jk}_{lh} = \hat{R}^{\mp 1ij}_{hl} g^{lk}. \quad (3.11)$$

The  $N$ -dimensional quantum Euclidean space is the associative algebra  $\mathbb{R}_q^N$  generated by elements  $\{x^i\}_{i=-n, \dots, n}$  with relations

$$\mathcal{P}_{a\,kl}^{ij} x^k x^l = 0. \quad (3.12)$$

These relations are preserved by the (right) action of the quantum group  $U_q so(N)$ , which is defined on the generators by

$$x^i \triangleleft g = \rho_j^i(g) x^j, \quad \rho_j^i(g) \in \mathbb{C}, \quad (3.13)$$

where  $\rho$  is the  $N$ -dimensional vector representation of  $U_q so(N)$ , and extended to the rest of  $\mathbb{R}_q^N$  so that the latter becomes a  $U_q so(N)$  module algebra. That is, for arbitrary  $g, g' \in U_q so(N)$  and  $a, a' \in \mathbb{R}_q^N$ , we have

$$a \triangleleft (gg') = (a \triangleleft g) \triangleleft g' \quad (3.14)$$

$$(aa') \triangleleft g = (a \triangleleft g_{(1)}) (a' \triangleleft g_{(2)}). \quad (3.15)$$

Here we have used Sweedler notation (with lower indices) for the coproduct,  $\Delta(g) = g_{(1)} \otimes g_{(2)}$ ; the right-hand side is actually a short-hand notation for a finite sum  $\sum_I g_{(1)}^I \otimes g_{(2)}^I$ .

Relations (3.12) can be written more explicitly in the form [30]

$$\begin{aligned} x^i x^j &= q x^j x^i && \text{for } i < j, i \neq -j, \\ [x^i, x^{-i}] &= k \omega_{i-1}^{-1} r_{i-1}^2 && \text{for } i > 1, \\ [x^1, x^{-1}] &= \begin{cases} 0 & \text{for } N \text{ even,} \\ h r_0^2 & \text{for } N \text{ odd.} \end{cases} \end{aligned} \quad (3.16)$$

We have here introduced  $h$  defined by

$$h \equiv q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad (3.17)$$

and we have defined as well a sequence of numbers  $r_i, r$  by

$$r_i^2 = \sum_{k,l=-i}^i g_{kl} x^k x^l, \quad r^2 \equiv r_n^2 \quad (3.18)$$

where  $i \geq 0$  for  $N$  odd, whereas for  $N$  even  $i \geq 1$  and of course in the sum only  $k, l \neq 0$  actually occur. The element  $r^2$  is  $SO_q(N)$ -invariant and generates the center of the algebra  $\mathbb{R}_q^N$ . It can be easily checked that

$$x^j r_i^2 = \begin{cases} r_i^2 x^j & \text{for } |j| \leq i, \\ q^2 r_i^2 x^j & \text{for } j < -i, \\ q^{-2} r_i^2 x^j & \text{for } j > i. \end{cases} \quad (3.19)$$

As this will be necessary for the construction of the elements  $\lambda_a$  be introduced in section 4, we now extend the algebra  $\mathbb{R}_q^N$  by adding the square root  $r_i$  of  $r_i^2$  for  $i = 0 \dots n$  as well as the inverses  $r_i^{-1}$  of these elements. As the relations (3.19) contain only  $q^{\pm 2}$  it is consistent to set for  $i \geq 0$

$$x^j r_i = \begin{cases} r_i x^j & \text{for } |j| \leq i, \\ q r_i x^j & \text{for } j < -i, \\ q^{-1} r_i x^j & \text{for } j > i. \end{cases} \quad (3.20)$$

We shall be mainly interested in the case  $q \in \mathbb{R}^+$ . In this case a conjugation

$$(x^i)^* = x^j g_{ji} \quad (3.21)$$

can be defined on  $\mathbb{R}_q^N$  to obtain what is known as real quantum Euclidean space. The elements  $r_i$  are then real.

There are [3] two differential calculi which are covariant with respect to  $U_q so(N)$ , obtained by imposing the condition

$$(d\alpha) \lhd g = d(\alpha \lhd g) \quad \alpha \in \Omega^*(\mathbb{R}_q^N) \quad (3.22)$$

on the differential. We denote the two exterior derivatives by  $d$  and  $\bar{d}$  and the corresponding exterior algebras by  $\Omega^*(\mathbb{R}_q^N)$  and  $\bar{\Omega}^*(\mathbb{R}_q^N)$ . If we introduce  $\xi^i = dx^i$  and  $\bar{\xi}^i = \bar{d}x^i$ , then they are characterized respectively by

$$x^i \xi^j = q \hat{R}_{kl}^{ij} \xi^k x^l, \quad (3.23)$$

$$x^i \bar{\xi}^j = q^{-1} \hat{R}^{-1ij} \bar{\xi}^k x^l. \quad (3.24)$$

For  $q \in \mathbb{R}^+$  neither  $\Omega^*(\mathbb{R}_q^N)$  nor  $\bar{\Omega}^*(\mathbb{R}_q^N)$  possesses an involution. However, one can introduce a  $*$ -structure on the direct sum  $\Omega^1(\mathbb{R}_q^N) \oplus \bar{\Omega}^1(\mathbb{R}_q^N)$  by setting

$$(\xi^i)^* = \bar{\xi}^j g_{ji}. \quad (3.25)$$

Using the properties (3.11, 3.3) of the  $\hat{R}$ -matrix one sees that the two calculi are conjugate; the Equations (3.23) and (3.24) are exchanged.

By taking the differential of (3.23) and (3.24) the  $\xi\xi$ -commutation relations are determined

$$\begin{aligned} \mathcal{P}_{s_{kl}}^{ij} \xi^k \xi^l &= 0, & \mathcal{P}_{t_{kl}}^{ij} \xi^k \xi^l &= 0, \\ \mathcal{P}_{s_{kl}}^{ij} \bar{\xi}^k \bar{\xi}^l &= 0, & \mathcal{P}_{t_{kl}}^{ij} \bar{\xi}^k \bar{\xi}^l &= 0. \end{aligned} \quad (3.26)$$

These relations define the algebraic structure of  $\Omega^*(\mathbb{R}_q^N)$  and  $\bar{\Omega}^*(\mathbb{R}_q^N)$ .

It is useful to introduce a set of gradings  $\deg_i$ ,  $i = 1, \dots, n$  on  $\Omega^*(\mathbb{R}_q^N)$  by

$$\deg_i(\xi^j) = \deg_i(x^j) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = -j, \\ 0 & \text{otherwise.} \end{cases} \quad (3.27)$$

All these gradings are preserved by the commutation relations (3.12), since the  $\hat{R}$ -matrix, and therefore any polynomial function of it like  $\mathcal{P}_a$ , fulfills (3.5). The  $n$ -ple  $(\deg_1, \dots, \deg_n)$  coincides with the weight vector of the fundamental vector representation of  $so(N)$ .

The Dirac operator [12], defined by Equation 2.3,

$$\xi^i = -[\theta, x^i] \quad (3.28)$$

is easily verified to be given by

$$\theta = \omega_n q^{\frac{N}{2}} k^{-1} r^{-2} g_{ij} x^i \xi^j, \quad (3.29)$$

as pointed out in [10, 35]. For the barred calculus  $\bar{\Omega}^*(\mathbb{R}_q^N)$  the ‘Dirac operator’  $\bar{\theta}$  (2.3) is

$$\bar{\theta} = -\omega_n q^{-\frac{N}{2}} k^{-1} r^{-2} g_{ij} x^i \bar{\xi}^j. \quad (3.30)$$

If  $q \in \mathbb{R}^+$  it satisfies

$$\theta^* = -\bar{\theta}. \quad (3.31)$$

In order to construct the  $\lambda_a$  and  $\theta^a$  satisfying the conditions described in Section 2 we first must solve the following problem. In Section 2 we assumed the center of the algebra  $\mathcal{A}$  to be trivial, which makes possible the construction of elements  $\lambda_a$  and  $\theta^a$  with the features described there. But the algebra generated by the  $x^i$  and  $r_j$  has a nontrivial center. With a general Ansatz of the type

$$\theta^a = \theta_i^a \xi^i \quad (3.32)$$

the condition  $[\theta^a, r_n^2] = 0$  can be rewritten as

$$(r_n^2 \theta_i^a - q^{-2} \theta_i^a r_n^2) \xi^i = 0, \quad (3.33)$$

which has no solution since  $r_n^2 \in \mathcal{Z}(\mathbb{R}_q^N)$ . To find a solution to (3.33) we further enlarge the algebra by adding a unitary element  $\Lambda$ , the ‘dilatator’, which satisfies the commutation relations

$$x^i \Lambda = q \Lambda x^i. \quad (3.34)$$

We also add its inverse  $\Lambda^{-1}$ . In the case  $N$  odd we can now follow the scheme previously proposed for  $N = 3$  [23] but in the case of even  $N$  the situation is slightly more complicated. We have added the elements  $r_1^{\pm 1} = (x^{-1} x^1)^{\pm \frac{1}{2}}$  and as a consequence the center is non trivial even after addition of  $\Lambda$ . The elements

$$r_1^{-1} x^{\pm 1} = (x^1 (x^{-1})^{-1})^{\pm \frac{1}{2}}$$

commute also with  $\Lambda$ . (We recall that  $x^{-1})^{-1}$  is the inverse of the element  $x^i$  with  $i = -1$ .) In other words, since the algebra generated by  $(\Lambda, r_1^{\pm 1}, x^{\pm 1}, \dots, x^{\pm n})$  is completely symmetric in the exchange of  $x^1$  and  $x^{-1}$ , there is no way to distinguish between these two elements. To have  $N$  linearly independent  $\theta^a$ , instead of fewer, we shall need to add yet another element to the algebra. We choose to add a ‘Drinfeld-Jimbo’ generator  $K = q^{\frac{H_1}{2}}$  and its inverse  $K^{-1}$ , where  $H_1$  belongs to the Cartan subalgebra

of  $U_q so(N)$  and represents the component of the angular momentum in the  $(-1, 1)$ -plane. This new element satisfies the commutation relations

$$\begin{aligned} Kx^{\pm 1} &= q^{\pm 1}x^{\pm 1}K, \\ Kx^{\pm i} &= x^{\pm i}K, \quad \text{for } i > 1, \end{aligned} \quad (3.35)$$

as well as

$$K\Lambda = \Lambda K. \quad (3.36)$$

When  $q \in \mathbb{R}^+$  it is compatible with the commutation relations to extend the  $*$ -structure (3.21)  $\Lambda, K$  as

$$\Lambda^* = \Lambda^{-1}, \quad K^* = K. \quad (3.37)$$

We must decide now which commutation relations  $\Lambda, K$  should satisfy with the  $\xi^i$ . As already observed [23] there are different possibilities.

A first possibility is to set [30]

$$\xi^i \Lambda = \Lambda \xi^i, \quad \Lambda d = q d \Lambda. \quad (3.38)$$

This choice has the disadvantage that  $\Lambda$  cannot be considered as an element of the quantum space, because due to (3.38) it does not satisfy the Leibniz rule  $d(fg) = f dg + (df)g \ \forall f, g \in \mathbb{R}_q^N$ . Nevertheless, it can be interpreted in a consistent way as an element of the Heisenberg algebra, because  $\Lambda^{-2}$  can be constructed [30] as a simple polynomial in the coordinates and derivatives.

Alternatively, what was considered also in [2], one could ask the Leibniz rule  $d(fg) = f dg + (df)g$  to hold also if  $f = \Lambda$ . By differentiating (3.34) one obtains that

$$\xi^i \Lambda + x^i d\Lambda = q d\Lambda x^i + q \Lambda \xi^i. \quad (3.39)$$

A solution would be to require that

$$x^i(d\Lambda) = q(d\Lambda)x^i, \quad \xi^i \Lambda = q \Lambda \xi^i. \quad (3.40)$$

In particular it would then be possible to set  $d\Lambda = \Lambda d$ , which implies that  $(d\Lambda) = 0$ . This choice is not completely satisfactory either since we would like the relation

$$df = 0 \text{ implies } f \propto 1 \quad (3.41)$$

to hold, and this would not be the case if  $d\Lambda = 0$ . As a consequence the general formalism is still not strictly applicable and there will be a conformal ambiguity in the choice of metric. We shall see below that with a procedure similar to the one described previously [23] for  $N = 3$ , we would recover  $\mathbb{R} \times S^{N-1}$  as geometry rather than  $\mathbb{R}^N$  in the commutative limit. Therefore, in the sequel we shall impose the first condition (3.38). As will be shown in the next section, this allows us to normalize the  $\theta^a$  and  $\lambda_a$  in such a way as to obtain  $\mathbb{R}^N$  as geometry in the commutative limit. The above discussion with  $\Lambda$  can be repeated to determine the

commutation relations between  $K$  and the 1-forms  $\xi^i$ . We choose  $dK = 0$ . Then consistency with (3.35) requires that

$$\begin{aligned} K\xi^{\pm 1} &= q^{\pm 1}\xi^{\pm 1}K, \\ K\xi^i &= \xi^iK, \quad \text{for } i > 1, \end{aligned} \quad (3.42)$$

To summarize, we shall consider the algebra  $\mathcal{A}_N$ , an extension of  $\mathbb{R}_q^N$  defined for odd  $N$  as

$$\mathcal{A}_N = \{x^i, r_j, r_j^{-1}, \Lambda, \Lambda^{-1} : -n \leq i \leq n, 0 \leq j < n\}$$

with generators which satisfy the relations (3.12), (3.20), (3.34) and for even  $N$  as

$$\mathcal{A}_N = \{x^i, r_j, r_j^{-1}, \Lambda, \Lambda^{-1}, K, K^{-1} : -n \leq i \leq n, 1 \leq j < n\}$$

with generators which satisfy the relations (3.12), (3.20), (3.34), (3.35). The algebra of differential forms  $\Omega^*(\mathcal{A}_N)$  is generated by the one-forms  $\xi^i$  satisfying relations (3.23), (3.26), (3.38) when  $N$  is odd, and (3.23), (3.26), (3.38), (3.42) when  $N$  is even. However one must bear in mind that the additional elements  $\Lambda$  and  $K$  are rather exceptional since  $dK = 0$  and either  $d\Lambda = 0$ , or it does not satisfy the Leibniz rule. These elements would be better interpreted as elements of the Heisenberg algebra.

## 4 Inner derivations and frame

We would like to construct a frame  $\theta^a$  and the associated inner derivations  $e_a = \text{ad } \lambda_a$  satisfying the conditions in Section 2 for the case of the algebra  $\mathcal{A}_N$ . We first solve this problem in a larger algebra, which we now define. It is possible to extend  $\Omega^*(\mathcal{A}_N)$  to the cross-product algebra  $\Omega^*(\mathcal{A}_N) \bowtie U_q so(N)$  by postulating the cross-commutation relations

$$\xi g = g_{(1)}(\xi \triangleleft g_{(2)}) \quad (4.1)$$

for any  $g \in U_q so(N)$  and  $\xi \in \Omega^*(\mathcal{A}_N)$ . The algebra  $\Omega^*(\mathcal{A}_N) \bowtie U_q so(N)$  can be made into a module algebra under the action  $\triangleleft$  of  $U_q so(N)$  by extending the latter on the elements of  $U_q so(N)$  as the adjoint action,

$$h \triangleleft g = Sg_{(1)}hg_{(2)}, \quad g, h \in U_q so(N) .$$

The  $S$  here denotes the antipode of  $U_q so(N)$ .

Let us introduce  $U_q^{op} so(N)$  the Hopf algebra with the same algebra structure of  $U_q so(N)$ , but opposite coalgebra, and by  $\Delta^{op}(g) = g_{(2)} \otimes g_{(1)}$  its coproduct. On any module algebra  $\mathcal{M}$  of  $U_q^{op} so(N)$  the corresponding action  $\overset{op}{\triangleleft}$  will thus fulfill the relations

$$a \overset{op}{\triangleleft} (gg') = (a \overset{op}{\triangleleft} g) \overset{op}{\triangleleft} g' \quad (4.2)$$

$$(aa') \overset{op}{\triangleleft} g = (a \overset{op}{\triangleleft} g_{(2)}) (a' \overset{op}{\triangleleft} g_{(1)}). \quad (4.3)$$

These are to be compared with (3.14) and (3.15). It is immediate to show that definition (4.1) implies that one can realize the action  $\triangleleft$  in the ‘adjoint-like way’

$$\eta \triangleleft g = Sg_{(1)} \eta g_{(2)} \quad (4.4)$$

on all of  $\Omega^*(\mathcal{A}_N) \bowtie U_q so(N)$ . On the other hand, one can realize also a corresponding action  $\overset{op}{\triangleleft}$  by

$$\eta \overset{op}{\triangleleft} g = (S^{-1}g_{(2)}) \eta g_{(1)}, \quad (4.5)$$

where  $S^{-1}$  is the antipode of  $\Delta^{op}$ .

We return now to the problem of the construction of a frame and of a set of dual inner derivations for the differential calculus  $(\Omega^*(\mathcal{A}_N), d)$ . As a first step, we must find  $N$  independent solutions  $\vartheta^a$  to the equation

$$[f, \vartheta^a] = 0 \quad \forall f \in \mathcal{A}_N. \quad (4.6)$$

We shall look first for solutions  $\vartheta^a$  in  $\Omega^*(\mathcal{A}_N) \bowtie U_q so(N)$ . The reason is the following. For each solution  $\vartheta^a$  of (4.6) and for any  $g \in U_q so(N)$  we can consider the image  $\vartheta_g^a \in \Omega^*(\mathcal{A}_N) \bowtie U_q so(N)$  of  $\vartheta^a$ , defined by

$$\vartheta_g^a := \vartheta^a \overset{op}{\triangleleft} g = (S^{-1}g_{(2)}) \vartheta^a g_{(1)}. \quad (4.7)$$

Now we show that its commutator with any element  $f \in \mathcal{A}_N$  vanishes:

$$\begin{aligned} [f, \vartheta_g^a] &= [f, (S^{-1}g)_{(1)} \vartheta^a S(S^{-1}g)_{(2)}] \\ &= f (S^{-1}g)_{(1)} \vartheta^a S(S^{-1}g)_{(2)} - (S^{-1}g)_{(1)} \vartheta^a S(S^{-1}g)_{(2)} f \\ &\stackrel{(4.1)}{=} (S^{-1}g)_{(1)} [f \triangleleft (S^{-1}g)_{(2)}] \vartheta^a S(S^{-1}g)_{(3)} \\ &\quad - (S^{-1}g)_{(1)} \vartheta^a [f \triangleleft (S^{-1}g)_{(2)}] S(S^{-1}g)_{(3)} \\ &= (S^{-1}g)_{(1)} [f \triangleleft (S^{-1}g)_{(2)}] \vartheta^a S(S^{-1}g)_{(3)} = 0 \end{aligned}$$

In the last equality we have used the fact that  $f \triangleleft (S^{-1}g)_{(2)} \in \mathcal{A}_N$  and (4.6). In other words  $\vartheta_g^a$  also behaves as a frame element. Moreover by its very definition  $\vartheta_g^a$  will in general belong to  $\Omega^1(\mathcal{A}_N) \bowtie U_q so(N)$  even if  $\vartheta^a \in \Omega^1(\mathcal{A}_N)$  (unless the  $n$ -plet  $\{\vartheta^a\}$  builds an irreducible representation of  $U_q^{op} so(N)$ ). We can summarize the above results in the

**Proposition 1** *The subspace  $\mathcal{F}$  of  $\Omega^1(\mathcal{A}_N) \bowtie U_q so(N)$  spanned by frame elements carries a representation of  $U_q^{op} so(N)$ .*

We shall call a set  $\{\vartheta^a\}_{a=-n, \dots, n}$  a generalized frame if  $\vartheta^a$  are elements of  $\Omega^1(\mathcal{A}_N) \bowtie U_q so(N)$  fulfilling (4.6), and any  $\xi \in \Omega^1(\mathcal{A}_N) \bowtie U_q so(N)$  can be uniquely decomposed in the form  $\vartheta^a \xi_a$  with  $\xi_a \in \mathcal{A}_N \bowtie U_q so(N)$ .

We now look for a generalized frame in the form of a basis of an irreducible  $N$ -dimensional representation of  $U_q^{op} so(N)$ . Recall that the universal  $R$ -matrix  $\mathcal{R}$  is a special element

$$\mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in U_q so(N) \otimes U_q so(N) \quad (4.8)$$

intertwining between  $\Delta$  and  $\Delta^{op}$ , and so does also  $\mathcal{R}_{21}^{-1}$ :

$$\mathcal{R} \Delta(\cdot) = \Delta^{op}(\cdot) \mathcal{R}, \quad \mathcal{R}_{21}^{-1} \Delta(\cdot) = \Delta^{op}(\cdot) \mathcal{R}_{21}^{-1}. \quad (4.9)$$

In (4.8) we have used a Sweedler notation with upper indices: the right-hand side is a short-hand notation for a sum  $\sum_I \mathcal{R}_I^{(1)} \otimes \mathcal{R}_I^{(2)}$  of infinitely many terms. The other main properties of  $\mathcal{R}$  are recalled in the appendix 7.2.

**Proposition 2** *Let  $(u_5)^{-1} = \mathcal{R}^{(1)}(S\mathcal{R}^{(2)})$ ,  $(u_7)^{-1} = \mathcal{R}^{-1(2)}(S\mathcal{R}^{-1(1)})$ . The elements of  $\Omega^1(\mathcal{A}_N) \rtimes U_q so(N)$*

$$\vartheta^a := \alpha u_7^{-1} \left( S\mathcal{R}^{(2)} \right) \xi^a \mathcal{R}^{(1)} \quad (4.10)$$

$$\bar{\vartheta}^a := \bar{\alpha} u_5^{-1} \left( S\mathcal{R}^{-1(1)} \right) \xi^a \mathcal{R}^{-1(2)} \quad (4.11)$$

are covariant under the action (4.5), more precisely

$$\vartheta^a \stackrel{op}{\triangleleft} g = \rho_b^a(g) \vartheta^b \quad \bar{\vartheta}^a \stackrel{op}{\triangleleft} g = \rho_b^a(g) \bar{\vartheta}^b. \quad (4.12)$$

In the previous definitions we have inserted two scalar factors  $\alpha, \bar{\alpha}$  to be fixed later.

**Proof** We prove the first formula. We find

$$\begin{aligned} \vartheta^a \stackrel{op}{\triangleleft} g &\stackrel{(4.5)}{=} (S^{-1}g_{(2)}) \vartheta^a g_{(1)} \stackrel{(4.10)}{=} (S^{-1}g_{(2)}) \alpha u_7^{-1} \left( S\mathcal{R}^{(2)} \right) \xi^a \mathcal{R}^{(1)} g_{(1)} \\ &\stackrel{(7.11)}{=} \alpha u_7^{-1} (Sg_{(2)}) \left( S\mathcal{R}^{(2)} \right) \xi^a \mathcal{R}^{(1)} g_{(1)} \\ &\stackrel{(4.9)}{=} \alpha u_7^{-1} \left( S\mathcal{R}^{(2)} \right) (Sg_{(1)}) \xi^a g_{(2)} \mathcal{R}^{(1)} \\ &\stackrel{(3.13)}{=} \rho_b^a(g) \alpha u_7^{-1} \left( S\mathcal{R}^{(2)} \right) \xi^b \mathcal{R}^{(1)} \stackrel{(4.10)}{=} \rho_b^a(g) \vartheta^b. \end{aligned}$$

The proof of the second formula is completely analogous.  $\square$

We can give an alternative and very useful expression for  $\vartheta^a, \bar{\vartheta}^a$ . It is convenient to introduce the Faddeev-Reshetikin-Taktadjan generators [19] of  $U_q so(N)$  :

$$\mathcal{L}_l^{+a} := \mathcal{R}^{(1)} \rho_l^a(\mathcal{R}^{(2)}) \quad \mathcal{L}_l^{-a} := \rho_l^a(\mathcal{R}^{-1(1)}) \mathcal{R}^{-1(2)}. \quad (4.13)$$

In our conventions

$$\mathcal{R} \in U_q^+ so(N) \otimes U_q^- so(N) \quad (4.14)$$

where  $U_q^+ so(N)$ ,  $U_q^- so(N)$  denote the positive and negative Borel subalgebras; hence we see that  $\mathcal{L}_l^{+a} \in U_q^+ so(N)$  and  $\mathcal{L}_l^{-a} \in U_q^- so(N)$ . From

formulae (7.8), (7.9) in the Appendix 7.2 one finds that the coproducts are given by

$$\Delta(\mathcal{L}_j^{+i}) = \mathcal{L}_h^{+i} \otimes \mathcal{L}_j^{+h} \quad \Delta(\mathcal{L}_j^{-i}) = \mathcal{L}_h^{-i} \otimes \mathcal{L}_j^{-h}. \quad (4.15)$$

Their commutation relations will be given in following section. Using them and (4.1) one easily proves the

**Proposition 3** *Let  $u_3 = \mathcal{R}^{(2)} S^{-1} \mathcal{R}^{(1)}$ ,  $u_4 = \mathcal{R}^{-1(1)} S^{-1} \mathcal{R}^{-1(2)}$ .*

$$\vartheta^a = \alpha \mathcal{L}_l^{-a} \eta^l = \alpha \eta^m \rho_m^l(u_4) \mathcal{L}_l^{-a} \quad (4.16)$$

$$\bar{\vartheta}^a = \bar{\alpha} \mathcal{L}_l^{+a} \eta^l = \bar{\alpha} \eta^m \rho_m^l(u_3) \mathcal{L}_l^{+a}, \quad (4.17)$$

with  $\eta^i = \xi^i$  or  $\eta^i = \bar{\xi}^i$ . Thus  $\vartheta^a$  and  $\bar{\vartheta}^a$  belong to  $\Omega^1(\mathcal{A}_N) \rtimes U_q^- so(N)$  and  $\Omega^1(\mathcal{A}_N) \rtimes U_q^+ so(N)$  respectively for  $\eta^i = \xi^i$ , or  $\bar{\Omega}^1(\mathcal{A}_N) \rtimes U_q^- so(N)$  and  $\bar{\Omega}^1(\mathcal{A}_N) \rtimes U_q^+ so(N)$  respectively for  $\eta^i = \bar{\xi}^i$ .

We now show that  $(\vartheta^a, \bar{\vartheta}^a)$  constitute a frame. Recall that the braid matrix can be written as  $\hat{R}_{hk}^{ij} = (\rho_h^j \otimes \rho_k^i) \mathcal{R}$ . Using (4.1), (4.15) one can easily prove

**Lemma 1**

$$x^i \mathcal{L}_b^{\pm a} = \mathcal{L}_c^{\pm a} x^j \hat{R}_{jb}^{\pm 1 ci}, \quad (4.18)$$

$$\xi^i \mathcal{L}_b^{\pm a} = \mathcal{L}_c^{\pm a} \xi^j \hat{R}_{jb}^{\pm 1 ci}, \quad (4.19)$$

$$\bar{\xi}^i \mathcal{L}_b^{\pm a} = \mathcal{L}_c^{\pm a} \bar{\xi}^j \hat{R}_{jb}^{\pm 1 ci}, \quad (4.20)$$

We should note that the commutation relations we have just found are different from (in a sense opposite to) the ones which we would have found by imposing, instead of (4.1), the condition

$$g\xi = g_{(1)} \triangleright \xi g_{(2)}, \quad (4.21)$$

with a *left* action  $\triangleright$ . This is true also in the commutative case ( $q = 1$ ), and in a sense is unpleasant since the latter definition of the commutation relations is what we are usually more familiar to. For instance, if  $h$  is a primitive generator in the Cartan subalgebra then  $[h, x^i]$  defined in the first way is opposite to the one defined in the second; or if  $g^+$  is a positive root, then  $[g^+, x^i]$  defined in the first way is proportional (up to a Cartan subalgebra factor) to the commutator  $[g^-, x^i]$  defined in the second way, where  $g^-$  is the negative root opposite to  $g^+$ .

The reason why we have adopted (4.1) is that this is necessary in order that the coordinates  $x^i$  carry *upper* indices (as it is conventional in general relativity) and that the representation  $\rho$  defined by (3.13) can be considered as the fundamental (vector) one, rather than its contragradient. This follows from  $\rho_h^i(gg') = \rho_j^i(g) \rho_h^j(g')$ . Had we used lower indices to label the coordinates, or replaced  $\rho(g)$  with  $\rho(Sg)$ , we could have adopted (4.21).

**Proposition 4** Assume  $\xi^i, \bar{\xi}^i$  are the differentials (3.23), (3.24), of the previous section. If we choose

$$\bar{\alpha} = \alpha^{-1} = \Lambda \quad (4.22)$$

then  $\{\vartheta^a\}, \{\bar{\vartheta}^a\}$  are generalized frames respectively in  $\Omega^1(\mathcal{A}_N) \rtimes U_q so(N)$  and  $\bar{\Omega}^1(\mathcal{A}_N) \rtimes U_q so(N)$ .

**Proof** We prove this in the first case.

$$x^i \vartheta^a \stackrel{(4.16)}{=} x^i \mathcal{L}^{-a}_l \Lambda^{-1} \xi^l \stackrel{(4.19)}{=} \mathcal{L}^{-a}_c \hat{R}^{-1}_{jl} x^j \Lambda^{-1} \xi^l \stackrel{(3.23)}{=} \mathcal{L}^{-a}_c \Lambda^{-1} \xi^c x^i \stackrel{(4.19)}{=} \vartheta^a x^i$$

The proof in the second case is completely analogous.  $\square$

In the commutative limit  $\{\xi^i\}_{i=1,\dots,n}$  is a frame and all other frames can be obtained from it by a  $G \subset U so(N)$  transformation. Formulae (4.16) and (4.17) say that in the noncommutative case one frame can be obtained from  $\{\xi^i\}$  by a very particular  $U_q so(N)$  transformation, the one with matrix elements  $\mathcal{L}^{\pm a}_b$ .

Note that by the choice (4.22)  $\vartheta^a, \bar{\vartheta}^a$  commute not only with the coordinates  $x^i, r_j$ , but also with the special element  $\Lambda$ ,

$$[\Lambda, \vartheta^a] = 0 \quad [\Lambda, \bar{\vartheta}^a] = 0, \quad (4.23)$$

under the assumption (3.38); had we assumed instead (3.40), then the same would be true by choosing  $\alpha = \Lambda^{-1} r$ ,  $\bar{\alpha} = \Lambda r$ . On the other hand, if we choose  $\bar{\alpha} = \bar{f}(r) \Lambda$ ,  $\alpha = f(r) \Lambda^{-1}$  (with  $f, \bar{f} \neq \text{const}$ ), then  $\vartheta^a, \bar{\vartheta}^a$  will still commute with the coordinates  $x^i, r_j$ , but in general not with  $\Lambda$ .

The commutation relations between the frame elements are given by the

**Proposition 5** The commutation relations among the  $\vartheta^a$  (resp.  $\bar{\vartheta}^a$ ) are as the ones among the  $\xi^i$  (resp.  $\bar{\xi}^i$ ), except for the opposite products:

$$\begin{aligned} \mathcal{P}_{s_{ab}}^{cd} \vartheta^b \vartheta^a &= 0, & \mathcal{P}_{t_{ab}}^{cd} \vartheta^b \vartheta^a &= 0 \\ \mathcal{P}_{s_{ab}}^{cd} \bar{\vartheta}^b \bar{\vartheta}^a &= 0, & \mathcal{P}_{t_{ab}}^{cd} \bar{\vartheta}^b \bar{\vartheta}^a &= 0. \end{aligned} \quad (4.24)$$

**Proof** We prove the claim in the first case (in the second the proof is completely analogous). For both  $\mathcal{P} = \mathcal{P}_s, \mathcal{P}_t$

$$\begin{aligned} \mathcal{P}_{ab}^{cd} \vartheta^b \vartheta^a &\stackrel{(4.16)}{=} \mathcal{P}_{cd}^{ab} \Lambda^{-1} \mathcal{L}^{-b}_l \xi^l \Lambda^{-1} \mathcal{L}^{-a}_m \xi^m \stackrel{(4.19)}{=} \Lambda^{-2} \mathcal{P}_{ab}^{cd} \mathcal{L}^{-b}_l \mathcal{L}^{-a}_h \xi^k \hat{R}^{-1}_{mk} \xi^m \\ &\stackrel{(5.6)}{=} \Lambda^{-2} \xi^l \xi^n (\mathcal{P} \hat{R}^{-1})^{mh}_{ln} \mathcal{L}^{-d}_h \mathcal{L}^{-c}_m \stackrel{(3.6)}{\propto} \Lambda^{-2} \xi^l \xi^n \mathcal{P}^{mh}_{ln} \mathcal{L}^{-d}_h \mathcal{L}^{-c}_m \stackrel{(3.26)}{=} 0. \end{aligned}$$

$\square$

We now decompose  $d$  (and similarly  $\bar{d}$ ) in terms both of differentials  $\xi^i = dx^i$  and  $U_q so(N)$ -covariant derivatives  $\partial_i$  and of frame elements  $\vartheta^a$  and derivations  $\epsilon_a$ :

$$d = \partial_i \xi^i = \epsilon_a \vartheta^a \quad d = \bar{\partial}_i \bar{\xi}^i = \bar{\epsilon}_a \bar{\vartheta}^a. \quad (4.25)$$

Looking at (4.16), (4.17) we find that

$$\partial_i = \epsilon_a \mathcal{L}^{-a}_i \Lambda^{-1} \quad \bar{\partial}_i = \bar{\epsilon}_a \mathcal{L}^{+a}_i \Lambda. \quad (4.26)$$

Now assume that the “Dirac operator” exists; it must necessarily be  $U_q so(N)$ -invariant. We can decompose it as well both in the basis of differentials and in the basis of frame elements:

$$\theta = -w_i \xi^i = -y_a \vartheta^a, \quad \bar{\theta} = -\bar{w}_i \bar{\xi}^i = -\bar{y}_a \bar{\vartheta}^a, \quad (4.27)$$

with  $w_i, \bar{w}_i \in \mathcal{A}_N$  and  $y_a \in \mathcal{A}_N \rtimes U_q^- so(N)$ ,  $\bar{y}_a \in \mathcal{A}_N \rtimes U_q^+ so(N)$ ; in this case

$$\epsilon_a = [y_a, \cdot] \quad \bar{\epsilon}_a = [\bar{y}_a, \cdot]. \quad (4.28)$$

The commutation relations among the  $w_i, \bar{w}_i$  will be of the form

$$\mathcal{P}_{aij}^{hk} w_k w_h = 0, \quad \mathcal{P}_{aij}^{hk} \bar{w}_k \bar{w}_h = 0. \quad (4.29)$$

From the  $U_q so(N)$  invariance of  $d, \bar{d}, \theta$  and  $\bar{\theta}$  it follows that

$$\mu_i \triangleleft g = \mu_l \rho_i^l(Sg), \quad \mu_i = w_i, \bar{w}_i, \partial_i, \bar{\partial}_i. \quad (4.30)$$

From (4.27), (4.16), (4.17) we find

$$y_a = w_i S \mathcal{L}^{-i}_a \Lambda \quad (4.31)$$

$$\bar{y}_a = \bar{w}_i S \mathcal{L}^{+i}_a \Lambda^{-1} \quad (4.32)$$

From the  $U_q^{op} so(N)$ -invariance of  $d, \bar{d}, \theta, \bar{\theta}$  we find the transformation rules

$$\nu_a \overset{op}{\triangleleft} g = \nu_b \rho_a^b(S^{-1}g), \quad \nu_a = y_a, \bar{y}_a, \epsilon_a, \bar{\epsilon}_a. \quad (4.33)$$

**Proposition 6** *The commutation relations among the  $y_a$  and among the  $\bar{y}_a$  are*

$$\mathcal{P}_{aab}^{cd} y_c y_d = 0, \quad \mathcal{P}_{aab}^{cd} \bar{y}_c \bar{y}_d = 0. \quad (4.34)$$

**Proof**

$$\begin{aligned} \mathcal{P}_{aab}^{cd} y_c y_d &\propto \mathcal{P}_{aab}^{cd} w_i (S \mathcal{L}^{-i}_c) w_j (S \mathcal{L}^{-j}_d) \Lambda^2 \\ &= \mathcal{P}_{aab}^{cd} w_i (w_j \triangleleft \mathcal{L}^{-i}_h) (S \mathcal{L}^{-h}_c) (S \mathcal{L}^{-j}_d) \Lambda^2 \\ &= \mathcal{P}_{aab}^{cd} w_i w_k \rho_j^k (S \mathcal{L}^{-i}_h) (S \mathcal{L}^{-h}_c) (S \mathcal{L}^{-j}_d) \Lambda^2 \\ &= w_i w_k \hat{R}_{hj}^{ki} S \left( \mathcal{L}^{-j}_d \mathcal{L}^{-h}_c \mathcal{P}_{aab}^{cd} \right) \Lambda^2 \\ &\stackrel{(5.6), (4.13)}{=} w_i w_k \hat{R}_{hj}^{ki} S \left( \mathcal{P}_{alm}^{hj} \mathcal{L}^{-m}_b \mathcal{L}^{-l}_a \right) \Lambda^2 \\ &\propto w_i w_k \mathcal{P}_{alm}^{ki} S \left( \mathcal{L}^{-m}_b \mathcal{L}^{-l}_a \right) \Lambda^2 \stackrel{(4.29)}{=} 0. \end{aligned}$$

The proof is completely analogous for  $\bar{y}_a$ .  $\square$

From  $\xi^i = [x^i, \theta] = [w_j \xi^j, x^i]$ ,  $\bar{\xi}^i = [x^i, \bar{\theta}] = [\bar{w}_j \bar{\xi}^j, x^i]$  it follows

$$x^i w_k \Lambda = \hat{R}^{-1ji}_{hk} w_j \Lambda x^h - \delta_h^i \Lambda \quad x^i \bar{w}_k \Lambda^{-1} = \hat{R}^{ji}_{hk} \bar{w}_j \bar{\Lambda} x^h - \delta_h^i \Lambda^{-1}$$

using these relations and (4.1), (4.15) one can easily prove

**Proposition 7**

$$[y_a, x^j] = \Lambda S \mathcal{L}^{-j}_a \quad [\bar{y}_a, x^j] = \Lambda^{-1} S \mathcal{L}^{+j}_a \quad (4.35)$$

These formulae can be seen as the inverse of (4.31), (4.32), since they give  $S \mathcal{L}^{+a}_j, S \mathcal{L}^{-a}_j$  in terms of  $y_a, \bar{y}_a$ .

In next section we find algebra homomorphisms

$$\varphi^+ : \mathcal{A}_N \rtimes U_q^+ so(N) \rightarrow \mathcal{A}_N, \quad (4.36)$$

$$\varphi^- : \mathcal{A}_N \rtimes U_q^- so(N) \rightarrow \mathcal{A}_N, \quad (4.37)$$

defined as the identity on  $\mathcal{A}_N$ ,

$$\varphi^\pm(a) = a \quad \text{if } a \in \mathcal{A}_N. \quad (4.38)$$

Then the 1-forms

$$\theta^a = \theta_l^a \xi^l \quad \theta_l^a := \varphi^-(\Lambda^{-1} \mathcal{L}^{-a}_l) \quad (4.39)$$

$$\bar{\theta}^a = \bar{\theta}_l^a \bar{\xi}^l \quad \bar{\theta}_l^a := \varphi^+(\Lambda \mathcal{L}^{+a}_l) \quad (4.40)$$

belong to  $\Omega^1(\mathcal{A}_N), \bar{\Omega}^1(\mathcal{A}_N)$  and still fulfil the property (4.6), in other words build up a frame. Similarly the elements of  $\mathcal{A}_N$  defined by

$$\lambda_a = \varphi^-(y_a) \quad \bar{\lambda}_a = \varphi^+(\bar{y}_a). \quad (4.41)$$

will yield the dual inner derivations,

$$e_a = [\lambda_a, \cdot] \quad \bar{e}_a = [\bar{\lambda}_a, \cdot] \quad (4.42)$$

It is clear that since  $\mathcal{Z}(\mathcal{A}_N) = \mathbb{C}$  then the frame and the dual set of inner derivations are *uniquely* determined up to a linear transformation (with numerical coefficients). By definition, the  $\lambda_a, \bar{\lambda}_a$  will still fulfill the commutation relations (4.34),

$$\mathcal{P}_{(a)cd}^{ab} \lambda_a \lambda_b = 0, \quad \mathcal{P}_{(a)cd}^{ab} \bar{\lambda}_a \bar{\lambda}_b = 0. \quad (4.43)$$

After application of  $\varphi^\pm$  equations (4.26) become

$$\partial_i = e_a \varphi^-(\mathcal{L}^{-a}_i) \Lambda^{-1} \quad \bar{\partial}_i = \bar{e}_a \varphi^+(\mathcal{L}^{+a}_i) \Lambda. \quad (4.44)$$

Their interest lies in the fact that they relate the  $SO_q(N)$ -covariant derivatives  $\partial_i, \bar{\partial}_i$  [3, 31], introduced following the approach of Woronowicz and Wess-Zumino [37, 38, 39, 36], to the inner derivations  $e_a, \bar{e}_a$  dual to the frame and which are defined using ordinary commutation relations, following the approach of Connes [11, 12].

On the other hand,  $\varphi^+, \varphi^-$  cannot be extended to homomorphisms of  $\Omega^*(\mathcal{A}_N) \rtimes U_q^\pm so(N) \rightarrow \Omega^*(\mathcal{A}_N)$ , since the commutation relations between the elements of  $U_q so(N)$  and the 1-forms do not map into the commutations of the elements of  $\mathcal{A}_N$  with the 1-forms; this is immediate to check if we use the frame basis: the frame elements don't commute with  $U_q^\pm so(N)$ , but do commute with  $\varphi^+(U_q^+ so(N)), \varphi^-(U_q^- so(N)) \subset \mathcal{A}_N$ . As a consequence, the commutation relations among the  $\theta_a, \bar{\theta}_a$  differ from (4.24) by a reversal of the product order. That is,

**Proposition 8** *The commutation relations among the  $\theta^a$  (resp.  $\bar{\theta}^a$ ) are as the ones among the  $\xi^i$  (resp.  $\bar{\xi}^i$ ):*

$$\begin{aligned} \mathcal{P}_{s_{ab}}^{cd} \theta^a \theta^b &= 0, & \mathcal{P}_{t_{ab}}^{cd} \theta^a \theta^b &= 0 \\ \mathcal{P}_{s_{ab}}^{cd} \bar{\theta}^a \bar{\theta}^b &= 0, & \mathcal{P}_{t_{ab}}^{cd} \bar{\theta}^a \bar{\theta}^b &= 0 \end{aligned} \quad (4.45)$$

**Proof** We prove the claim in the first case. For both  $\mathcal{P} = \mathcal{P}_s, \mathcal{P}_t$  we have

$$\begin{aligned} \mathcal{P}_{ab}^{cd} \theta^a \theta^b &\stackrel{(4.39)}{=} \mathcal{P}_{ab}^{cd} \theta^a \theta_n^b \xi^n \stackrel{(4.6)}{=} \mathcal{P}_{ab}^{cd} \theta_n^b \theta^a \xi^n \\ &\stackrel{(4.39)}{=} \mathcal{P}_{ab}^{cd} \Lambda^{-2} \varphi^-(\mathcal{L}_n^{-b}) \varphi^-(\mathcal{L}_l^{-a}) \xi^l \xi^n \propto \Lambda^{-2} \varphi^-(\mathcal{P}_{ab}^{cd} \mathcal{L}_n^{-b} \mathcal{L}_l^{-a}) \xi^l \xi^n \\ &\stackrel{(5.6)}{=} \Lambda^{-2} \varphi^-(\mathcal{L}_b^{-d} \mathcal{L}_a^{-c} \mathcal{P}_{ln}^{ab}) \xi^l \xi^n \stackrel{(3.26)}{=} 0 \end{aligned}$$

The proof is completely analogous for  $\bar{y}_a$ .  $\square$

The  $\theta^a, \bar{\theta}^a, \lambda_a, \bar{\lambda}_a$  do not inherit from  $\vartheta^a, \bar{\vartheta}^a, y_a, \bar{y}_a$  the transformation properties (4.12), (4.33) under the action  $\overset{op}{\triangleleft}$  of  $U_q^{op} so(N)$ . For the  $\theta^a, \bar{\theta}^a$  this is clear since the commutation relations (4.45) are no longer compatible with the action of  $U_q^{op} so(N)$ . As a consequence, this is true also for the  $\lambda_a, \bar{\lambda}_a$ , because  $\theta = -\theta^a \lambda_a, \bar{\theta} = -\bar{\theta}^a \bar{\lambda}_a$  are invariant.

One could in principle define an  $U_q^{op} so(N)$  action  $\overset{op'}{\triangleleft}$  on  $\mathcal{A}_N$  by postulating instead

$$\lambda_a \overset{op'}{\triangleleft} g = \lambda_b \rho_a^b (S^{-1} g) \quad \bar{\lambda}_a \overset{op'}{\triangleleft} g = \bar{\lambda}_b \rho_a^b (S^{-1} g);$$

but the latter would differ from the action  $\overset{op}{\triangleleft}$  fulfilling (4.5). We shall therefore not do so.

## 5 Homomorphism $\mathcal{A}_N \rtimes U_q so(N) \rightarrow \mathcal{A}_N$

It is known [19] that a set of generators of  $U_q so(N)$  is provided by  $\{\mathcal{L}_j^{+i}, \mathcal{L}_j^{-i}\}$  and some further elements obtained by introducing square roots and inverses of the elements  $\mathcal{L}_i^{\pm i}$ , which is always possible because they are invertible elements belonging to the Cartan subalgebra. The set  $\{\mathcal{L}_j^{+i}, \mathcal{L}_j^{-i}\}$  has  $2N^2$  elements, but  $N(N-1)$  of them vanish due to the upper and lower triangularity of the matrices  $\mathcal{L}^{\pm}$ , whereas their diagonal elements are the inverses of each other:

$$\mathcal{L}_j^{+i} = 0, \quad \text{if } i > j \quad (5.1)$$

$$\mathcal{L}_j^{-i} = 0, \quad \text{if } i < j \quad (5.2)$$

$$\mathcal{L}_i^{+i} \mathcal{L}_i^{-i} = 1, \quad \forall i \quad (5.3)$$

$$\mathcal{L}_{-n}^{\pm n} \dots \mathcal{L}_n^{\pm n} = 1. \quad (5.4)$$

To these relations we have to add the relations characterizing  $U_q so(N)$ ,

$$\mathcal{L}_j^{\pm i} \mathcal{L}_k^{\pm h} g^{kj} = g^{hi} \quad \mathcal{L}_i^{\pm j} \mathcal{L}_h^{\pm k} g_{kj} = g_{hi}, \quad (5.5)$$

which further reduce to  $N(N-1)/2$  the number of independent generators, and finally the commutation relations

$$\hat{R}_{cd}^{ab} \mathcal{L}_f^{+d} \mathcal{L}_e^{+c} = \mathcal{L}_c^{+b} \mathcal{L}_d^{+a} \hat{R}_{ef}^{dc} \quad (5.6)$$

$$\hat{R}_{cd}^{ab} \mathcal{L}_f^{-d} \mathcal{L}_e^{-c} = \mathcal{L}_c^{-b} \mathcal{L}_d^{-a} \hat{R}_{ef}^{dc} \quad (5.7)$$

$$\hat{R}_{cd}^{ab} \mathcal{L}_f^{+d} \mathcal{L}_e^{-c} = \mathcal{L}_c^{-b} \mathcal{L}_d^{+a} \hat{R}_{ef}^{dc}. \quad (5.8)$$

Note that (5.4) yields no constraint for  $N$  even since it is a consequence of (5.5), whereas for  $N$  odd it yields one further constraint. In fact (5.5) implies that  $(\mathcal{L}^{+0})^2 = 1$  which together with (5.4) implies that  $\mathcal{L}_0^{+0} = 1$ . The antipode on the generators takes the form

$$S\mathcal{L}_j^{\pm i} = g^{hi} \mathcal{L}_h^{\pm k} g_{jk} \quad (5.9)$$

To construct a homomorphism  $\varphi : \mathcal{A}_N \rtimes U_q so(N) \rightarrow \mathcal{A}_N$  acting as the identity on  $\mathcal{A}_N$  it is therefore sufficient to define it on  $\mathcal{L}_j^{+i}, \mathcal{L}_j^{-i}$  and to verify that all of relations (4.19) and (5.1) to (5.8) are satisfied. Applying  $\varphi$  to (4.35) and using (5.9) it follows that

$$\varphi(\mathcal{L}_j^{-i}) \stackrel{(5.9)}{=} g^{ih} \varphi(S\mathcal{L}_h^{-k}) g_{kj} \stackrel{(4.35)}{=} g^{ih} \Lambda^{-1}[\lambda_h, x^k] g_{kj} \quad (5.10)$$

$$\varphi(\mathcal{L}_j^{+i}) \stackrel{(5.9)}{=} g^{ih} \varphi(S\mathcal{L}_h^{+k}) g_{kj} \stackrel{(4.35)}{=} g^{ih} \Lambda[\bar{\lambda}_h, x^k] g_{kj}. \quad (5.11)$$

The problem is thus equivalent to the construction of  $N$  objects  $\lambda_a$  and  $N$  objects  $\bar{\lambda}_a$  such that relations (5.10), (5.11) define elements  $\varphi(\mathcal{L}_h^{\pm k})$  in  $\mathcal{A}_N$  which satisfy all of relations (4.19)<sub>1</sub> and (5.1) to (5.8).

Actually for the construction of a frame and a set of dual inner derivations in  $\Omega^*(\mathcal{A}_N)$  (resp.  $\bar{\Omega}^*(\mathcal{A}_N)$ ) we just need a homomorphism  $\varphi^- : U_q^- so(N) \ltimes \mathcal{A}_N \rightarrow \mathcal{A}_N$  (resp.  $\varphi^+ : U_q^+ so(N) \ltimes \mathcal{A}_N \rightarrow \mathcal{A}_N$ ) acting as the identity on  $\mathcal{A}_N$ , what we look for first. This is equivalent to looking for  $N$  objects  $\lambda_a$  (resp.  $\bar{\lambda}_a$ ) such that the objects  $\varphi^-(\mathcal{L}_j^{-i}) = g^{ih} \Lambda^{-1}[\lambda_h, x^k] g_{kj}$  (resp.  $\varphi^+(\mathcal{L}_j^{+i}) = g^{ih} \Lambda[\bar{\lambda}_h, x^k] g_{kj}$ ) fulfill just the relations involving only  $\mathcal{L}^-$  (resp.  $\mathcal{L}^+$ ). Finally we find which further conditions  $\lambda_a, \bar{\lambda}_a$  must fulfill in order that we can ‘glue’  $\varphi^-, \varphi^+$  into a unique homomorphism  $\varphi$ .

**Theorem 1** *One can define a homomorphism  $\varphi^- : \mathcal{A}_N \rtimes U_q^- so(N) \rightarrow \mathcal{A}_N$  by setting on the generators*

$$\varphi^-(a) = a, \quad \forall a \in \mathcal{A}_N, \quad (5.12)$$

$$\varphi^-(\mathcal{L}_j^{-i}) = g^{ih} \Lambda^{-1}[\lambda_h, x^k] g_{kj}, \quad (5.13)$$

with

$$\begin{aligned} \lambda_0 &= \gamma_0 \Lambda(x^0)^{-1} && \text{for } N \text{ odd,} \\ \lambda_{\pm 1} &= \gamma_{\pm 1} \Lambda(x^{\pm 1})^{-1} K^{\mp 1} && \text{for } N \text{ even,} \\ \lambda_a &= \gamma_a \Lambda r_{|a|}^{-1} r_{|a|-1}^{-1} x^{-a} && \text{otherwise,} \end{aligned} \quad (5.14)$$

and  $\gamma_a \in \mathbb{C}$  normalization constants fulfilling the conditions

$$\begin{aligned} \gamma_0 &= -q^{-\frac{1}{2}} h^{-1} && \text{for } N \text{ odd,} \\ \gamma_1 \gamma_{-1} &= \begin{cases} -q^{-1} h^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} && (5.15) \\ \gamma_a \gamma_{-a} &= -q^{-1} k^{-2} \omega_a \omega_{a-1} && \text{for } a > 1. \end{aligned}$$

The proof is given in Appendix 7.3. The relation (5.15) fixes only the product  $\gamma_a \gamma_{-a}$ . Notice that  $\gamma_0^2$  for  $N$  odd and  $\gamma_1 \gamma_{-1}$  for  $N$  even are positive real numbers, while all the remaining products  $\gamma_a \gamma_{-a}$  are negative. Notice that the embedding  $\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+2}$  defined by (7.4) automatically induces an embedding for the corresponding  $\lambda_a$ . Similarly one can prove

**Theorem 2** *One can define a homomorphism  $\mathcal{A}_N \rtimes U_q^+ so(N) \rightarrow \mathcal{A}_N$  by setting on the generators*

$$\varphi^+(a) = a, \quad \forall a \in \mathcal{A}_N, \quad (5.16)$$

$$\varphi^+(\mathcal{L}_j^{+i}) = g^{ih} \Lambda[\bar{\lambda}_h, x^k] g_{kj}, \quad (5.17)$$

with

$$\begin{aligned} \bar{\lambda}_0 &= \bar{\gamma}_0 \Lambda^{-1}(x^0)^{-1} && \text{for } N \text{ odd,} \\ \bar{\lambda}_{\pm 1} &= \bar{\gamma}_{\pm 1} \Lambda^{-1}(x^{\pm 1})^{-1} K^{\pm 1} && \text{for } N \text{ even,} \\ \bar{\lambda}_a &= \bar{\gamma}_a \Lambda^{-1} r_{|a|}^{-1} r_{|a|-1}^{-1} x^{-a} && \text{otherwise,} \end{aligned} \quad (5.18)$$

and  $\bar{\gamma}_a \in \mathbb{C}$  normalization constants fulfilling the conditions

$$\begin{aligned}\bar{\gamma}_0 &= q^{\frac{1}{2}} h^{-1} && \text{for } N \text{ odd,} \\ \bar{\gamma}_1 \bar{\gamma}_{-1} &= \begin{cases} -qh^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} && (5.19) \\ \bar{\gamma}_a \bar{\gamma}_{-a} &= -qk^{-2} \omega_a \omega_{a-1} && \text{for } a > 1.\end{aligned}$$

To ‘glue’  $\varphi^+, \varphi^-$  into a unique homomorphism  $\varphi : \mathcal{A}_N \rtimes U_q so(N) \rightarrow \mathcal{A}_N$  we still need to satisfy the image under  $\varphi$  of equations (5.8) and (5.3). As we shall prove in appendix 7.5, this is possible only in the case of odd  $N$  and completely fixes the coefficients  $\gamma_a, \bar{\gamma}_a$  in the previous Ansätze:

**Theorem 3** *In the case of odd  $N$  we one can define a homomorphism  $U_q so(N) \rtimes \mathcal{A}_N \rightarrow \mathcal{A}_N$  by setting on the generators*

$$\varphi(a) = a \quad \forall a \in \mathcal{A}_N \quad (5.20)$$

$$\varphi(\mathcal{L}_j^{-i}) = g^{ih} \Lambda^{-1}[\lambda_h, x^k] g_{kj}, \quad (5.21)$$

$$\varphi(\mathcal{L}_j^{+i}) = g^{ih} \Lambda[\bar{\lambda}_h, x^k] g_{kj}, \quad (5.22)$$

with  $\lambda_j, \bar{\lambda}_j$  defined as in (5.14), (5.18) and with coefficients given by

$$\begin{aligned}\gamma_0 &= -q^{-\frac{1}{2}} h^{-1} \\ \gamma_1^2 &= -q^{-2} h^{-2} \\ \gamma_a^2 &= -q^{-2} \omega_a \omega_{a-1} k^{-2} && \text{for } a > 1 \\ \gamma_a &= q\gamma_{-a} && \text{for } a \leq 1 \\ \bar{\gamma}_a &= -q\gamma_a.\end{aligned} \quad (5.23)$$

Notice that the  $\gamma_a, \bar{\gamma}_a$  for  $a \neq 0$  are imaginary and fixed only up to a sign. This has as a consequence that the homomorphism  $\varphi$  does not preserve the star structure of  $U_q so(N)$ , in other words it is not a star-homomorphism. In the case  $N = 3$  this is the same homomorphism which is also constructed in [6]. Alternatively, we can fix the normalizations of the barred objects so that for  $q \in \mathbb{R}^+$  the involution gives

$$\lambda_a^* = -g^{ab} \bar{\lambda}_b, \quad (\theta^a)^* = \bar{\theta}^b g_{ba}. \quad (5.24)$$

Then the coefficients  $\bar{\gamma}_a$  will be related to the  $\gamma_a$  by

$$\begin{aligned}\bar{\gamma}_0 &= -q\gamma_0^*, && \text{if } N \text{ odd,} \\ \bar{\gamma}_{\pm 1} &= -\gamma_{\mp 1}^*, && \text{if } N \text{ even,} \\ \bar{\gamma}_a &= -\gamma_{-a}^* \begin{cases} 1 & \text{if } a > 0 \\ q^2 & \text{if } a < 0 \end{cases} && \text{otherwise.}\end{aligned} \quad (5.25)$$

We summarize our results for the frames  $\theta^a, \bar{\theta}^a$ :

$$\theta^a = \theta_l^a \xi^l = \Lambda^{-2} g^{ab} [\lambda_b, x^j] g_{jl} \xi^l \quad (5.26)$$

$$\bar{\theta}^a = \bar{\theta}_l^a \bar{\xi}^l = \Lambda^2 g^{ab} [\bar{\lambda}_b, x^j] g_{jl} \bar{\xi}^l. \quad (5.27)$$

These objects commute both with  $x^i, r_j$  and with  $\Lambda$ . Had we adopted the commutation rule (3.40), instead of (3.38), then the same would be true by introducing an additional factor  $r$  in the right-hand side of both the previous equations. The matrix elements  $\bar{\theta}_i^a, \theta_i^a$  fulfill the  $\varphi^+, \varphi^-$  images of equation (5.6), (5.7)

$$\hat{R}_{cd}^{ab} \theta_j^d \theta_i^c = \theta_l^b \theta_k^a \hat{R}_{ij}^{kl} \quad \hat{R}_{cd}^{ab} \bar{\theta}_j^d \bar{\theta}_i^c = \bar{\theta}_l^b \bar{\theta}_k^a \hat{R}_{ij}^{kl}. \quad (5.28)$$

## 6 Metrics and linear connections on the quantum Euclidean space

In this Section we construct the covariant derivative, the corresponding linear connection and the metric associated with the frames introduced in Section 5. We follow for  $\mathbb{R}_q^N$  the same scheme proposed previously [23] for  $\mathbb{R}_q^3$ .

Since the  $\Omega^1(\mathcal{A}_N)$  and  $\bar{\Omega}^1(\mathcal{A}_N)$ , we recall, are free modules, the covariant derivatives can be defined by their actions on the frame

$$\begin{aligned} D\theta^a &= -\omega_{bc}^a \theta^b \otimes \theta^c, \\ \bar{D}\bar{\theta}^a &= -\bar{\omega}_{bc}^a \bar{\theta}^b \otimes \bar{\theta}^c. \end{aligned} \quad (6.1)$$

For the generalized permutation  $\sigma$  we can write

$$\sigma(\theta^a \otimes \theta^b) = S_{cd}^{ab} \theta^c \otimes \theta^d. \quad (6.2)$$

For the same reasons as for the coefficients of the metric the requirement of bilinearity

$$f S_{cd}^{ab} \theta^c \otimes \theta^d = \sigma(f \theta^a \otimes \theta^b) = \sigma(\theta^a \otimes \theta^b f) = S_{cd}^{ab} f \theta^c \otimes \theta^d \quad (6.3)$$

forces  $S_{cd}^{ab} \in \mathcal{Z}(\mathcal{A}_N) = \mathbb{C}$ . According to the consistency condition (2.15) for the torsion and the commutation relations (4.45) for the frame we find that the matrix  $S$  has the form

$$S = C_s \mathcal{P}_s - \mathcal{P}_a + C_t \mathcal{P}_t, \quad (6.4)$$

with  $C_s$  and  $C_t$  complex  $N^2 \times N^2$  matrices. As the next step we would like to define the metric according to (2.8).

$$g(\theta^a \otimes \theta^b) = g^{ab}. \quad (6.5)$$

But we have a problem. Due to the form of (3.6) it is not possible to satisfy simultaneously the metric compatibility condition and (6.4). Similarly to what was previously done for  $N = 3$  [23] the best we can do is to weaken the compatibility condition to a condition of proportionality. In this way like in the case  $N = 3$  we find the two solutions

$$S = q\hat{R}, \quad S = (q\hat{R})^{-1}, \quad (6.6)$$

corresponding to the choices  $C_s = q^2$ ,  $C_t = q^{2-N}$  or  $C_s = q^{-2}$ ,  $C_t = q^{N-2}$  respectively. As  $\hat{R}$  does, also both these solutions for  $S$  satisfy the Yang-Baxter equation (3.1). Now, using the property (3.11) for the  $\hat{R}$ -matrix, we see that

$$S_{df}^{ae} g^{fg} S_{eg}^{cb} = q^{\pm 2} g^{ac} \delta_d^b. \quad (6.7)$$

This has to be compared with the metric compatibility condition (2.21), which in a basis becomes

$$S_{df}^{ae} g^{fg} S_{eg}^{cb} = g^{ac} \delta_d^b. \quad (6.8)$$

The metric is in fact compatible with the linear connection only up to a conformal factor.

We can compute the action of  $\sigma$  on the basis  $\xi^i \otimes \xi^j$

$$\sigma(\xi^i \otimes \xi^j) = S_{hk}^{ij} \xi^h \xi^k. \quad (6.9)$$

To see this we have started with the definition (6.2) of the action on  $\theta^a \otimes \theta^b$ , and used (3.32). Then the result follows from (5.28). Notice that (6.9) coincides with (6.2). In a similar way, from (6.5), (3.32) and (7.24) we can determine the action of  $g$  on the  $\xi$ .

$$g(\xi^i \otimes \xi^j) = g^{ij} \Lambda^2. \quad (6.10)$$

This expression contains  $\Lambda$ , therefore the addition of this element to the algebra is a necessary condition for the construction of the metric, even if we would perform the calculation directly in the basis given by  $\xi^i$ . According to (2.16) a covariant derivative can be defined by

$$D\xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta). \quad (6.11)$$

As  $\theta$  is  $SO_q(N)$ -invariant, so is  $D$ . This is true for both choices of  $\sigma$ . One can show [23] that the expression (6.11) is the most general torsion-free,  $SO_q(N)$ -invariant linear connection.

The explicit action of the covariant derivative on  $\xi^i$  can be computed. We apply (3.11), (3.23) and obtain for  $S = q^{-1}\hat{R}^{-1}$

$$D\xi^i = 0. \quad (6.12)$$

Analogously, from (3.11), (3.23), (7.3) it turns out that

$$\begin{aligned}
D\xi^i &= -\theta \otimes \xi^i + q^{2+\frac{N}{2}}\omega_n k^{-1} r^{-2} g_{hj} (\hat{R}^2)_{lm}^{ji} x^h \xi^l \otimes \xi^m \\
&= (q^2 - 1)\theta \otimes \xi^i - q^2 \omega_n r^{-2} g_{lm} \left( q^{1-\frac{N}{2}} x^i \xi^l \otimes \xi^m + q^{\frac{N}{2}} \hat{R}_{hj}^{mi} x^l \xi^h \otimes \xi^j \right) \\
&= (q^2 - 1)(\theta \otimes \xi^i + \xi^i \otimes \theta) - q^{3-\frac{N}{2}} \omega_n r^{-2} x^i \xi^l \otimes \xi^m g_{lm}. \tag{6.13}
\end{aligned}$$

The coordinates are well adapted to the first linear connection, but not to the second one, because in the latter case  $D\xi^i \neq 0$ .

However, for both of these covariant derivatives the corresponding curvature (2.23) vanishes.

$$\text{Curv}(\xi) = 0 \tag{6.14}$$

This can be seen by performing the same calculation done previously [23] for  $N = 3$ .

$$\begin{aligned}
\text{Curv}(\xi) &= \pi_{12}\sigma_{012}\sigma_{023}\sigma_{012}(\theta^a \otimes \theta^b \otimes \theta^c) \xi_a \lambda_b \lambda_c \\
&= \pi_{12}(S_{12}S_{23}S_{12})^{abc}_{def}(\theta^d \otimes \theta^e \otimes \theta^f) \xi_a \lambda_b \lambda_c \\
&= (S_{12}S_{23}S_{12})^{abc}_{def}(\theta^d \theta^e \otimes \theta^f) \xi_a \lambda_b \lambda_c \\
&= -(S_{12}S_{23}\mathcal{P}_{a12})^{abc}_{def}(\theta^d \theta^e \otimes \theta^f) \xi_a \lambda_b \lambda_c \\
&= -(\mathcal{P}_{a23}S_{12}S_{23})^{abc}_{def}(\theta^d \theta^e \otimes \theta^f) \xi_a \lambda_b \lambda_c \\
&= 0.
\end{aligned}$$

We have used here the braid relation (3.1) for  $S$ , (2.15) and (4.43).

To conclude this section we repeat the above construction for the calculus  $\bar{\Omega}^1(\mathcal{A}_N)$ . Again, there is a unique metric  $g$  and two torsion-free  $SO_q(N)$ -covariant linear connections compatible with it up to a conformal factor. In the  $\bar{\theta}^a$  basis the actions of  $g$  and  $\sigma$  are respectively

$$g(\bar{\theta}^a \otimes \bar{\theta}^b) = g^{ab} \tag{6.15}$$

$$\sigma(\bar{\theta}^a \otimes \bar{\theta}^b) = \bar{S}^{ab}_{cd} \bar{\theta}^c \otimes \bar{\theta}^d, \tag{6.16}$$

The two choices for  $\sigma$  are

$$\bar{S} = q\hat{R}, \quad \text{or} \quad \bar{S} = (q\hat{R})^{-1}. \tag{6.17}$$

This implies

$$\bar{S}^{ae}_{df} g^{fg} \bar{S}^{cb}_{eg} = q^{\pm 2} g^{ac} \delta_d^b. \tag{6.18}$$

In the  $\bar{\xi}^i$  basis the actions of  $g$  and  $\sigma$  become

$$g(\bar{\xi}^i \otimes \bar{\xi}^j) = g^{ij} \Lambda^{-2}, \tag{6.19}$$

$$\sigma(\bar{\xi}^i \otimes \bar{\xi}^j) = \bar{S}^{ij}_{hk} \bar{\xi}^h \otimes \bar{\xi}^k. \tag{6.20}$$

The two covariant derivatives, one for each choice of  $\sigma$ , are

$$\bar{D}\bar{\xi} = -\bar{\theta} \otimes \bar{\xi} + \sigma(\bar{\xi} \otimes \bar{\theta}). \quad (6.21)$$

The associated linear curvatures  $\overline{\text{Curv}}$  vanish.

Since in the commutative limit  $q \rightarrow 1$  we have

$$\Lambda \rightarrow 1, \quad K \rightarrow 1,$$

we have also

$$g(\xi^i \otimes \xi^j) \rightarrow \delta^{i,-j} \quad g(\bar{\xi}^i \otimes \bar{\xi}^j) \rightarrow \delta^{i,-j}.$$

The right-hand side is the matrix of coefficients of the flat metric in the complex cartesian coordinates  $x^i$ , and we recover  $\mathbb{R}^N$  as geometry (at least formally). Had we adopted the commutation rule (3.40), instead of (3.38), then we would have found an additional factor  $\lim_{q \rightarrow 1} r^2$  in the right-hand side, which would have corresponded to the coefficients of the metric of  $\mathbb{R} \times S^{N-1}$ .

## 7 Appendix

### 7.1 Miscellanea

In this appendix we give some miscellanea formulae on  $\mathbb{R}_q^N$ .

The braid matrix of  $SO_q(N)$  is given by [19]

$$\begin{aligned} \hat{R} = & q \sum_{i \neq 0} \delta_i^i \otimes \delta_i^i + \sum_{\substack{i \neq j, -j \\ \text{or } i=j=0}} \delta_i^j \otimes \delta_j^i + q^{-1} \sum_{i \neq 0} \delta_i^{-i} \otimes \delta_{-i}^i \\ & + k \left( \sum_{i < j} \delta_i^i \otimes \delta_j^j - \sum_{i < j} q^{-\rho_i + \rho_j} \delta_i^{-j} \otimes \delta_{-i}^j \right) \end{aligned} \quad (7.1)$$

where here  $\delta_j^i$  is the  $N \times N$  matrix with all elements equal to zero except for a 1 in the  $i$ th column and  $j$ th row. Clearly,  $R_{hk}^{ij} := \hat{R}_{hk}^{ji}$  is a lower-triangular matrix. Moreover, under substitution of  $q$  by  $q^{-1}$  in (7.1) we find that

$$\hat{R}(q^{-1})_{kl}^{ij} = \hat{R}(q)^{-1-i-j}_{-k-l}. \quad (7.2)$$

Using the projector decomposition (3.6) and the expression (3.8) for  $\mathcal{P}_t$  the square of the  $\hat{R}$ -matrix can be computed to be

$$(\hat{R}^2)_{kl}^{ij} = k \hat{R}_{kl}^{ij} + \delta_k^i \delta_l^j - q^{1-N} k g^{ij} g_{kl}. \quad (7.3)$$

There is an embedding  $\mathcal{A}_N \hookrightarrow \mathcal{A}_{N+2}$  given by

$$\begin{aligned} x^i &\rightarrow x^i & \text{for } -n \leq i \leq n, \\ r_i &\rightarrow r_i & \text{for } 0 \leq i \leq n \\ \Lambda &\rightarrow \Lambda, \quad K \rightarrow K. \end{aligned} \quad (7.4)$$

It follows from (3.16) that one can rewrite (3.18) as

$$r_i^2 = \omega_i (q^{-1} \omega_{i-1}^{-1} r_{i-1}^2 + x^i x^{-i}) = \omega_i (q \omega_{i-1}^{-1} r_{i-1}^2 + x^{-i} x^i), \quad (7.5)$$

for  $i > 1$  and for  $N$  odd and  $i = 1$ . This implies that for  $i \geq 1$

$$x^{-i} x^i - q^2 x^i x^{-i} = -q k \omega_i^{-1} r_i^2. \quad (7.6)$$

Finally, we recall a useful property of the fundamental representation  $\rho$  of  $U_q so(N)$ , namely

$$\rho_b^a(Sg) = g^{ad} \rho_d^c(g) g_{cb}. \quad (7.7)$$

## 7.2 Universal $R$ -matrix

In this appendix we recall the basics about the universal  $R$ -matrix [15] of a quantum group  $U_q \mathfrak{g}$ , while fixing our conventions. We recall some useful formulae

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23} \quad (7.8)$$

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12} \quad (7.9)$$

$$(S \otimes \text{id})\mathcal{R} = \mathcal{R}^{-1} = (\text{id} \otimes S^{-1})\mathcal{R} \quad (7.10)$$

$$S^{-1}(g) = u^{-1} S(g) u. \quad (7.11)$$

Here  $u$  is any of the elements  $u_1, u_2, \dots, u_8$  defined below:

$$\begin{aligned} u_1 &:= (S\mathcal{R}^{(2)})\mathcal{R}^{(1)} & u_2 &:= (S\mathcal{R}^{-1(1)})\mathcal{R}^{-1(2)} \\ u_3 &:= \mathcal{R}^{(2)} S^{-1} \mathcal{R}^{(1)} & u_4 &:= \mathcal{R}^{-1(1)} S^{-1} \mathcal{R}^{-1(2)} \\ (u_5)^{-1} &:= \mathcal{R}^{(1)} S\mathcal{R}^{(2)} & (u_6)^{-1} &:= (S^{-1} \mathcal{R}^{(1)})\mathcal{R}^{(2)} \\ (u_7)^{-1} &:= \mathcal{R}^{-1(2)} S\mathcal{R}^{-1(1)} & (u_8)^{-1} &:= (S^{-1} \mathcal{R}^{-1(2)})\mathcal{R}^{-1(1)} \end{aligned} \quad (7.12)$$

In fact, using the results of Drinfel'd [15, 16] one can show that

$$u_1 = u_3 = u_7 = u_8 = v u_2 = v u_4 = v u_5 = v u_6, \quad (7.13)$$

where  $v$  is a suitable element belonging to the center of  $U_q so(N)$ .

From (4.9) and (7.8, 7.9) it follows the universal Yang-Baxter relation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}, \quad (7.14)$$

whence the other two relations follow

$$\mathcal{R}^{-1}_{12} \mathcal{R}^{-1}_{13} \mathcal{R}^{-1}_{23} = \mathcal{R}^{-1}_{23} \mathcal{R}^{-1}_{13} \mathcal{R}^{-1}_{12} \quad (7.15)$$

$$\mathcal{R}_{13} \mathcal{R}_{23} \mathcal{R}^{-1}_{12} = \mathcal{R}^{-1}_{12} \mathcal{R}_{23} \mathcal{R}_{13} \quad (7.16)$$

By applying  $\text{id} \otimes \rho_c^a \otimes \rho_d^b$  to (7.14),  $\rho_c^a \otimes \rho_d^b \otimes \text{id}$  to (7.15) and  $\rho_c^a \otimes \text{id} \otimes \rho_d^b$  to (7.16) we respectively find the commutation relations (5.6), (5.7), (5.8).

### 7.3 Proof of Theorem 1

We divide the proof in various steps. First note that, because of (3.11), equation (4.19) for  $\varphi^-(\mathcal{L}^-)$  is equivalent to

$$x^h[\lambda_a, x^i] = q\hat{R}_{jk}^{hi}[\lambda_a, x^j]x^k. \quad (7.17)$$

**Proposition 9**  *$N$  independent solutions to Equation (7.17) are given by (5.14) where  $\gamma_a \in \mathbb{C}$  are arbitrary normalization constants.*

**Proof** As a first step in the proof that Equation (7.17) is satisfied by the  $\lambda_a$  of (5.14), we calculate the commutation relations between the  $x^i$  and the  $\lambda_a$ :

$$\begin{aligned} \Lambda\varphi(\mathcal{L}^{-\frac{a}{i}}) &= q^{\rho_a - \rho_i}[\lambda_a, x^i] = 0 && \text{for } i < a, \\ \Lambda\varphi(\mathcal{L}^{-\frac{a}{i}}) &= q^{\rho_a - \rho_i}[\lambda_a, x^i] = -q^{\rho_a - \rho_i + 1}k\lambda_a x^i && \text{for } i > a, \\ \Lambda\varphi(\mathcal{L}^{-\frac{a}{a}}) &= [\lambda_a, x^a] && = \begin{cases} -\gamma_a \Lambda q k \omega_a^{-1} r_{a-1}^{-1} r_a & \text{for } a > 1, \\ \gamma_a \Lambda k \omega_{|a|-1}^{-1} r_{|a|}^{-1} r_{|a|-1} & \text{for } a < -1, \end{cases} \\ \Lambda\varphi(\mathcal{L}^{-\frac{0}{0}}) &= [\lambda_0, x^0] && = -q^{\frac{1}{2}} h \gamma_0 \Lambda && \text{for } N \text{ odd} \\ \Lambda\varphi(\mathcal{L}^{-\frac{1}{-1}}) &= [\lambda_1, x^1] && = \begin{cases} -q k \lambda_1 x^1 & \text{for } N \text{ even}, \\ -\gamma_1 \Lambda q h (x^0)^{-1} r_1 & \text{for } N \text{ odd}, \end{cases} \\ \Lambda\varphi(\mathcal{L}^{-\frac{1}{1}}) &= [\lambda_{-1}, x^{-1}] && = \begin{cases} -q k \lambda_{-1} x^{-1} & \text{for } N \text{ even}, \\ \gamma_{-1} \Lambda h (r_1)^{-1} x^0 & \text{for } N \text{ odd}, \end{cases} \end{aligned} \quad (7.18)$$

To obtain these relations we have used (3.20), (3.34) and (3.16). In the case  $a = i$  we also need (7.6), and for even  $N$ ,  $|a| = 1$  (3.35). It will be noticed that although complicated in appearance the system of Equations (7.18) is actually mainly the first two equations, which are quite simple, plus a series of special cases when  $i = a$ . The commutation relations between the  $x^i$  and the  $\lambda_a$  are independent of the normalization of the latter, so that they impose no restriction on the  $\gamma_a$ .

Writing down the explicit expression for the  $\hat{R}$ -matrix (7.1) and using the fact that  $[\lambda_a, x^i] = 0$  for  $i < a$ , one finds that the relation (7.17) becomes:

$$\begin{aligned} x^h[\lambda_a, x^i] &= q[\lambda_a, x^i]x^h + kq[\lambda_a, x^h]x^i && \text{for } h < i, h \neq -i \\ x^h[\lambda_a, x^i] &= q[\lambda_a, x^i]x^h && \text{for } \begin{cases} h > i, h \neq -i, \\ h = i = 0; \end{cases} \\ x^i[\lambda_a, x^i] &= q^2[\lambda_a, x^i]x^i && \text{for } h = i \neq 0; \end{aligned} \quad (7.19)$$

finally, when  $h = -i$ ,

$$\begin{aligned} x^{-i}[\lambda_a, x^i] &= [\lambda_a, x^i]x^{-i} + \\ &\quad kq \left( [\lambda_a, x^{-i}]x^i - \sum_{k < i} q^{-\rho_k + \rho_i} [\lambda_a, x^k]x^{-k} \right), \quad \text{for } i > 0, \\ x^{-i}[\lambda_a, x^i] &= [\lambda_a, x^i]x^{-i} - kq \sum_{k < i} q^{-\rho_k + \rho_i} [\lambda_a, x^k]x^{-k} \quad \text{for } i < 0. \end{aligned}$$

These relations can be checked one by one with a lengthy but straightward calculation by substituting the explicit expression (7.18) for  $[\lambda_a, x^i]$  and commuting  $x^h$  through it. More particularly, in the case  $i < a$  both sides of the equations are identically 0, because both  $[\lambda_a, x^i]$  and  $R = P\hat{R}$  are lower-triangular matrices. In the case  $i > a$  we first use (7.18) again to commute  $x^h$  with  $\lambda_a$ . Then we need (3.16) to commute  $x^h$  with  $x^i$  if  $h \neq -i$ , while we need to apply (7.5) to the expressions of the type  $x^k x^{-k}$  to write them in terms of  $r_{|k|}^2$  and  $r_{|k|-1}^2$  if  $h = -i$ . In the case  $i = a$  we need (3.20) to commute  $x^h$  through  $r_{|a|}$  and  $r_{|a|-1}$ . In the particular case  $i = h = 0$  (7.19) follows from the  $\Lambda x$  commutation relation (3.34).  $\square$

Next, look for  $\gamma_a$  such that equations (5.5), (5.4) for  $\varphi^-(\mathcal{L}^-)$  with  $i = -h$ ,

$$\varphi^-(\mathcal{L}^{-i}_i \mathcal{L}^{-i}_{-i}) = 1, \quad \varphi^-(\mathcal{L}^{-0}_0) = 1, \quad (7.20)$$

are fulfilled. Using (7.18), we easily find the  $\gamma_a$ 's given in equations (5.15).

**Lemma 2** *With the  $\gamma_a$  given in equations (5.15) the elements  $\lambda_a$  are also solutions to equations (4.43) and equations*

$$\lambda_a[\lambda_b, x^i] = q^{-1}(\hat{R}^{-1})_{ab}^{cd}[\lambda_c, x^i]\lambda_d. \quad (7.21)$$

We shall occasionally use the short-hand notations

$$e_a^i := [\lambda_a, x^i] \quad \bar{e}_a^i := [\bar{\lambda}_a, x^i] \quad (7.22)$$

Note that, because of (3.11), the  $\varphi^-$  images of equations (5.7) and (5.5) are respectively equivalent to the ‘RTT-relations’

$$\hat{R}_{kl}^{ij} e_a^k e_b^l = e_c^i e_d^j \hat{R}_{ab}^{cd} \quad (7.23)$$

and the ‘gTT-relations’

$$g^{ab} e_a^i e_b^j = g^{ij} \Lambda^2, \quad g_{ij} e_a^i e_b^j = g_{ab} \Lambda^2. \quad (7.24)$$

**Proposition 10** *With the  $\gamma_a$  given in equations (5.15) the matrices  $e_a^i$  fulfill (7.23), (7.24), and the  $\varphi^-$  image of (5.4).*

This will conclude the proof of Theorem 1.

**Proof of Lemma 2** It is interesting to note that the commutation relations (4.43) between the  $\lambda_a$  are the same as those (3.12) satisfied by the  $x^i$ , because  $\mathcal{P}_{acd}^{ab} = \mathcal{P}_{ab}^{cd}$ . As (3.12) is equivalent to (3.16), so will equations (4.43) be equivalent to

$$\begin{aligned} \lambda_a \lambda_b &= q \lambda_b \lambda_a && \text{for } a < b, a \neq -b, \\ [\lambda_a, \lambda_{-a}] &= k \omega_{a-1}^{-1} s_{a-1}^2 && \text{for } a > 1, \\ [\lambda_1, \lambda_{-1}] &= \begin{cases} 0 & \text{for } N \text{ even,} \\ h s_0^2 & \text{for } N \text{ odd,} \end{cases} \end{aligned} \quad (7.25)$$

where the quantities  $s_a^2$  are defined by the equation

$$s_a^2 = \sum_{c,d=-a}^a g^{cd} \lambda_c \lambda_d \quad (7.26)$$

for  $a \geq 0$  in the case  $N$  odd, and for  $a \geq 1$  in the case  $N$  even (in the latter case the sum of course runs over  $c, d \neq 0$ ).

It is easy to show the commutation relations (for  $i \geq 0$ )

$$r_i \lambda_a = \begin{cases} q^2 \lambda_a r_i & \text{for } a < -i, \\ q \lambda_a r_i & \text{for } |a| \leq i, \\ \lambda_a r_i & \text{for } a > i, \end{cases} \quad (7.27)$$

$$\lambda_a \Lambda = q^{-1} \Lambda \lambda_a, \quad (7.28)$$

and, for  $N$  even,

$$\begin{aligned} [K, \lambda_b] &= 0 && \text{for } |b| \neq 1, \\ K \lambda_{\pm 1} &= q^{\mp 1} \lambda_{\pm 1} K. \end{aligned} \quad (7.29)$$

which follow from (3.36).

To show now the relation (4.43)<sub>1</sub> we consider first the case  $a < b$ , excluding the cases  $N$  odd and  $a = 0$ , and  $N$  even and  $a = \pm 1$ . By using (7.18)<sub>2</sub>, (7.27) and (7.28) we obtain respectively the identities

$$\lambda_a \lambda_b = \gamma_a \Lambda r_{|a|}^{-1} r_{|a|-1}^{-1} \lambda_b x^{-a} = \gamma_a \Lambda \lambda_b r_{|a|}^{-1} r_{|a|-1}^{-1} x^{-a} = q \lambda_b \lambda_a.$$

If  $N$  is even and  $a = \pm 1$  we obtain

$$\lambda_{\pm 1} \lambda_b = \gamma_{\pm 1} \Lambda (x^{\pm 1})^{-1} \lambda_b K^{\mp 1} = \gamma_1 \Lambda \lambda_b (x^{\pm 1})^{-1} K^{\mp 1} = q \lambda_b \lambda_{\pm 1}, \quad (7.30)$$

using respectively the identities (7.29), (7.18) and (7.28). We proceed similarly in the case  $|b| = 1$  when  $N$  is even. The calculation is analogous

for the other cases  $b \neq -a$ . Summing up, for  $b \neq -a$  the  $\lambda_a$  and  $\lambda_b$   $q$ -commute, so that there is no restriction on the normalization constants  $\gamma_a$ .

We now consider the cases  $a = -b$ . It follows from (5.14), (5.15) that

$$\begin{aligned} s_0^2 &= \Lambda^2 q^{-2} h^{-2} (x_0)^{-2} && \text{for } N \text{ odd} \\ s_1^2 &= \Lambda^2 q^{-2} k^{-2} \omega_1^2 r_1^{-2} && \text{for } N \text{ even.} \end{aligned} \quad (7.31)$$

We use these two relations as initial steps to show by induction that

$$s_a^2 = q^{-2} \Lambda^2 \omega_a^2 k^{-2} r_a^{-2} \quad \text{for } a \geq 1. \quad (7.32)$$

In fact

$$\begin{aligned} s_a^2 &\stackrel{(7.26)}{=} s_{a-1}^2 + q^{-\rho_a} \lambda_a \lambda_{-a} + q^{\rho_a} \lambda_{-a} \lambda_a \\ &\stackrel{(5.14)}{=} s_{a-1}^2 + \Lambda^2 \gamma_a \gamma_{-a} r_a^{-2} r_{a-1}^{-2} [q^{-2-\rho_a} x^{-a} x^a + q^{\rho_a} x^a x^{-a}] \\ &\stackrel{(7.5)}{=} s_{a-1}^2 + \Lambda^2 \gamma_a \gamma_{-a} r_a^{-2} r_{a-1}^{-2} [q^{-2-\rho_a} (\omega_a^{-1} r_a^2 - q \omega_{a-1}^{-1} r_{a-1}^2) \\ &\quad + q^{\rho_a} (\omega_a^{-1} r_a^2 - q^{-1} \omega_{a-1}^{-1} r_{a-1}^2)] \\ &= s_{a-1}^2 + \Lambda^2 \gamma_a \gamma_{-a} r_a^{-2} r_{a-1}^{-2} [q^{-1} \omega_{a-1} \omega_a^{-1} r_a^2 - q^{-1} \omega_a \omega_{a-1}^{-1} r_{a-1}^2] \\ &\stackrel{(5.15)}{=} s_{a-1}^2 - \Lambda^2 q^{-2} k^{-2} \omega_{a-1}^2 r_{a-1}^{-2} + \Lambda^2 q^{-2} k^{-2} \omega_a^2 r_a^{-2}. \end{aligned}$$

Assuming that (7.32) holds for  $a = b - 1$ , the first two terms in the last line are opposite and therefore cancel, and the third gives (7.32) for  $a = b$ , as claimed. We now consider the commutators  $[\lambda_a, \lambda_{-a}]$  with  $a \geq 1$ . For  $N$  even we find the claim  $[\lambda_1, \lambda_{-1}] = 0$  by a straightforward calculation. In all other cases we proceed as follows,

$$\begin{aligned} [\lambda_a, \lambda_{-a}] &\stackrel{(5.14), (3.20)}{=} \gamma_a \gamma_{-a} q^{-1} \Lambda^2 r_a^{-2} r_{a-1}^{-2} (q^{-1} x^{-a} x^a - q x^a x^{-a}) \\ &\stackrel{(7.6)}{=} -q^{-1} \Lambda^2 \gamma_a \gamma_{-a} \omega_a^{-1} k r_{a-1}^{-2} \stackrel{(5.15)}{=} q^{-2} k^{-1} \omega_{a-1} \Lambda^2 r_{a-1}^{-2} \\ &\stackrel{(7.32)}{=} k \omega_{a-1}^{-1} s_{a-1}^2, \end{aligned}$$

as claimed.

With the choice (5.15) for the normalization constants  $\gamma_a$ , the algebra generated by  $x^i, \lambda_i, \Lambda^{\pm 1}, K^{\pm 1}, r_i^{\pm 1}$  is symmetric, i.e. is invariant, with

respect to the following transformation  $\mathcal{S}$

$$\begin{aligned}
\lambda_{\pm 1} &\rightarrow (\gamma_{\pm 1}(q^{-1})\gamma_{\mp 1}(q)^{-1})^{\frac{1}{2}} x^{\mp 1} && \text{for } N \text{ even,} \\
\lambda_a &\rightarrow q^{-\frac{1}{2}} (\gamma_a(q^{-1})\gamma_{-a}(q)^{-1})^{\frac{1}{2}} x^{-a} && \text{otherwise,} \\
x^{\pm 1} &\rightarrow (\gamma_{\pm 1}(q^{-1})\gamma_{\mp 1}(q)^{-1})^{\frac{1}{2}} \lambda_{\mp 1} && \text{for } N \text{ even,} \\
x^a &\rightarrow q^{-\frac{1}{2}} (\gamma_a(q^{-1})\gamma_{-a}(q)^{-1})^{\frac{1}{2}} \lambda_{-a} && \text{otherwise,} \\
\Lambda &\rightarrow \Lambda, \\
K &\rightarrow K^{-1}, \\
q &\rightarrow q^{-1}.
\end{aligned} \tag{7.33}$$

Notice that  $\mathcal{S}$  is an involution,  $\mathcal{S}^2 = \text{id}$ . Because of (3.3), (7.2) and (3.5) under  $\mathcal{S}$  the  $xx$ -commutation relations (3.12) and the  $\lambda\lambda$ -commutation relations (4.43) are exchanged. The  $x^a\Lambda$  (3.34) and the  $\lambda_a\Lambda$  relations (7.28) are exchanged as well, while the  $x\lambda$  relations (7.18) are invariant. We can immediately check that for  $N$  odd under  $\mathcal{S}$

$$r_a^2 \rightarrow \sum_{b=-a}^a q^{\rho_b} q^{-1} \lambda_{-b} \lambda_b (\gamma_b(q^{-1})\gamma_{-b}(q)^{-1})^{\frac{1}{2}} = s_a^2 \tag{7.34}$$

where we have used (5.15). In the case of even  $N$  the same calculation holds for the terms with  $|b| > 1$ , but we have to treat the term with  $|b| = 1$  separately

$$r_1^2 \rightarrow 2\lambda_{-1}\lambda_1(\gamma_1(q^{-1})\gamma_{-1}(q)^{-1})^{\frac{1}{2}} = s_1^2 \tag{7.35}$$

The relation (5.14) expressing  $\lambda_a$  in terms of  $x^{-a}$  is invariant as well. To see this we first express  $r_a^{-1}$  and  $r_{a-1}^{-1}$  in (5.14) through  $s_a$  and  $s_{a-1}$  respectively, then we use (7.28) to move  $\Lambda^{-1}$  to the left. In this way we are able to rewrite (5.14) in the form

$$\lambda_a = \gamma_a(q) q^2 \Lambda^{-1} s_a s_{a-1} x^{-a}. \tag{7.36}$$

Taking into account (5.15) and (3.34), under  $\mathcal{S}$  (7.36) becomes

$$x^{-a} = \gamma_a(q)^{-1} r_a r_{a-1} \Lambda^{-1} \lambda_a \tag{7.37}$$

i.e. we recover (5.14). Again, in the case of even  $N$  the special case  $|a| = 1$  has to be treated separately, but due to (7.33)<sub>6</sub> and (7.35), it is easily checked that (5.14) is invariant in this case, too.

This transformation is useful, because it enables us to get (7.21) by applying  $\mathcal{S}$  to (7.17) and then using the properties (3.3), (7.2) and (3.5) of the  $\hat{R}$ -matrix and (5.15). For  $N$  odd and  $N$  even,  $h \neq -i$

$$\begin{aligned}
& \lambda_h[x^a, \lambda_i] \\
&= \sum_{j,k} q^{-1} \hat{R}(q^{-1})_{-j-k}^{-h-i} \sqrt{\frac{\gamma_h(q)}{\gamma_{-h}(q^{-1})} \frac{\gamma_i(q)}{\gamma_{-i}(q^{-1})} \frac{\gamma_{-j}(q^{-1})}{\gamma_j(q)} \frac{\gamma_{-k}(q^{-1})}{\gamma_k(q)}} [x^a, \lambda_j] \lambda_k \\
&= \sum_{j,k} q^{-1} \hat{R}(q)^{-1jk}_{hi} [x^a, \lambda_j] \lambda_k.
\end{aligned} \tag{7.38}$$

In the particular case that  $N$  is even and  $h = -i$  from

$$\begin{aligned}
\gamma_1(q) \gamma_{-1}(q) &= \gamma_1(q^{-1}) \gamma_{-1}(q^{-1}) \\
\gamma_b(q) \gamma_{-b}(q) &= q^{-2} \gamma_b(q^{-1}) \gamma_{-b}(q^{-1}) \quad \text{for } b \neq \pm 1
\end{aligned}$$

and the property (3.5) of the  $\hat{R}$ -matrix, it is easily seen that (7.21) still holds. This concludes the proof of Lemma 2.  $\square$

**Proof of Proposition 10** To prove (7.23), we use (7.21) and (7.17)

$$\begin{aligned}
& \hat{R}_{ab}^{cd}[\lambda_c, x^i][\lambda_d, x^j] = \hat{R}_{ab}^{cd}(\lambda_c x^i - x^i \lambda_c)[\lambda_d, x^j] = \\
& \hat{R}_{ab}^{cd}(q \hat{R}_{kl}^{ij} \lambda_c[\lambda_d, x^k] x^l - q^{-1} (\hat{R}^{-1})_{cd}^{ef} x^i[\lambda_e, x^j] \lambda_f) = \\
& \hat{R}_{ab}^{cd} q^{-1} (\hat{R}^{-1})_{cd}^{ef} q \hat{R}_{kl}^{ij}[\lambda_e, x^k](\lambda_f x^l - x^l \lambda_f) = \hat{R}_{kl}^{ij}[\lambda_a, x^k][\lambda_b, x^l],
\end{aligned}$$

i.e. the ‘RTT’-relations for  $e_a^i$ . By repeated application of the ‘RTT’-relations it is an immediate result that for any polynomial  $f(\hat{R})$

$$f(\hat{R}_{kl}^{ij}) e_a^k e_b^l = e_c^i e_d^j f(\hat{R}_{ab}^{cd}). \tag{7.39}$$

In particular the projectors  $\mathcal{P}_s$ ,  $\mathcal{P}_a$ ,  $\mathcal{P}_t$  are of this form. If we write  $\mathcal{P}_t$  explicitly using (3.8), this yields the  $gTT$ -relations (7.24) [which are equivalent to the  $\varphi^-$ -image of equations (5.5)] also for  $h \neq -i$ , which we had not proved yet.  $\square$

## 7.4 Proof of Theorem 2

The proof of 2 is similar to the one of Theorem 1. The explicit expressions for  $\Lambda^{-1}\varphi^+(\mathcal{L}^{+-a}_{-i})$  are

$$\begin{aligned}
\Lambda^{-1}\varphi(\mathcal{L}^{+-a}_{-i}) &= q^{\rho_a - \rho_i} [\bar{\lambda}_a, x^i] = 0 && \text{for } i > a, \\
\Lambda^{-1}\varphi(\mathcal{L}^{+-a}_{-i}) &= q^{\rho_a - \rho_i} [\bar{\lambda}_a, x^i] = q^{\rho_a - \rho_i - 1} k \bar{\lambda}_a x^i && \text{for } i < a, \\
\Lambda^{-1}\varphi(\mathcal{L}^{+-a}_{-a}) &= [\bar{\lambda}_a, x^a] && = \begin{cases} -\bar{\gamma}_a \Lambda^{-1} k \omega_{a-1}^{-1} r_a^{-1} r_{a-1} & \text{for } a > 1, \\ \bar{\gamma}_a \Lambda^{-1} k q^{-1} \omega_a^{-1} r_{|a|-1}^{-1} r_{|a|} & \text{for } a < -1, \end{cases} \\
\Lambda^{-1}\varphi(\mathcal{L}^{+0}_0) &= [\bar{\lambda}_0, x^0] && = \bar{\gamma}_0 \Lambda^{-1} q^{-\frac{1}{2}} h && \text{for } N \text{ odd} \\
\Lambda^{-1}\varphi(\mathcal{L}^{+-1}_{-1}) &= [\bar{\lambda}_1, x^1] && = \begin{cases} q^{-1} k \bar{\lambda}_1 x^1 & \text{for } N \text{ even,} \\ -\bar{\gamma}_1 \Lambda^{-1} h r_1^{-1} x^0 & \text{for } N \text{ odd,} \end{cases} \\
\Lambda^{-1}\varphi(\mathcal{L}^{+1}_1) &= [\bar{\lambda}_{-1}, x^{-1}] && = \begin{cases} q^{-1} k \bar{\lambda}_{-1} x^{-1} & \text{for } N \text{ even,} \\ \bar{\gamma}_{-1} \Lambda^{-1} h q^{-1} r_1 (x^0)^{-1} & \text{for } N \text{ odd.} \end{cases}
\end{aligned} \tag{7.40}$$

## 7.5 Proof of Theorem 3

Using relations (3.11), it is easy to show that the image under  $\varphi$  of (5.8) is equivalent to

$$\hat{R}_{ab}^{cd} \bar{e}_c^i e_d^j = \hat{R}_{kl}^{ij} e_a^k \bar{e}_b^l \tag{7.41}$$

in the notation (7.22).

Theorem 1 (2) fixes the coefficients  $\gamma_a$  ( $\bar{\gamma}_a$ ) for  $a < 0$  in terms of  $\gamma_a$  ( $\bar{\gamma}_a$ ) for  $a \geq 0$ . We can use the remaining freedom in the choice of  $\gamma_a$ ,  $\bar{\gamma}_a$  to find further conditions on  $\gamma_a$  for  $a > 0$ , and relations relating the coefficients  $\bar{\gamma}_a$  to  $\gamma_a$  so that (5.3) and (5.8) are fulfilled. We start with the observation that in the case of odd  $N$  (5.14) and (5.18) imply

$$\bar{\lambda}_a = \bar{\gamma}_a \gamma_a^{-1} \Lambda^{-2} \lambda_a \tag{7.42}$$

and use the equations (7.21) and (7.17). In this way we see that

$$\begin{aligned}
\sum_{c,d} \hat{R}_{ab}^{cd} [\bar{\lambda}_c, x^i] [\lambda_d, x^j] &= \sum_{c,d} \hat{R}_{ab}^{cd} \bar{\gamma}_c \gamma_c^{-1} (\Lambda^{-2} \lambda_c x^i - x^i \Lambda^{-2} \lambda_c) [\lambda_d, x^j] = \\
\sum_{c,d} \hat{R}_{ab}^{cd} \bar{\gamma}_c \gamma_c^{-1} \Lambda^{-2} (\lambda_c x^i - q^{-2} x^i \lambda_c) [\lambda_d, x^j] &= \\
\sum_{c,d} \hat{R}_{ab}^{cd} \bar{\gamma}_c \gamma_c^{-1} \Lambda^{-2} (q \sum_{k,l} \hat{R}_{kl}^{ij} \lambda_c [\lambda_d, x^k] x^l - q^{-3} \sum_{e,f} \hat{R}_{cd}^{-1ef} x^i [\lambda_e, x^j] \lambda_f) &= \\
\sum_{c,d} \sum_{e,f} \sum_{k,l} \hat{R}_{ab}^{cd} \hat{R}_{cd}^{-1ef} \hat{R}_{kl}^{ij} \bar{\gamma}_c \gamma_c^{-1} [\lambda_e, x^k] (\Lambda^{-2} \lambda_f x^l - x^l \Lambda^{-2} \lambda_f) &= \\
\sum_{c,d} \sum_{e,f} \sum_{k,l} \hat{R}_{ab}^{cd} \hat{R}_{cd}^{-1ef} \hat{R}_{kl}^{ij} \bar{\gamma}_c \gamma_c^{-1} \gamma_f \bar{\gamma}_f^{-1} [\lambda_e, x^k] [\bar{\lambda}_f, x^l], &
\end{aligned}$$

If the further condition holds,

$$\bar{\gamma}_a \gamma_a^{-1} = c \equiv \text{const}, \quad (7.43)$$

then in the last line the  $\gamma$ 's cancel with each other, so do  $\hat{R}$  and  $\hat{R}^{-1}$ , and we find that (7.41) is actually satisfied.

To see that (7.43) is not only a sufficient, but also a necessary condition for (7.41), we write down the latter for the particular values of the indices  $i = a, j = b = a + 1$  for  $a = -n, \dots, n - 1$ .

$$\begin{aligned} & [\bar{\lambda}_{a+1}, x^a][\lambda_a, x^{a+1}] + k[\bar{\lambda}_a, x^a][\lambda_{a+1}, x^{a+1}] = \\ & [\lambda_a, x^{a+1}][\bar{\lambda}_{a+1}, x^a] + k[\lambda_a, x^a][\bar{\lambda}_{a+1}, x^{a+1}] \end{aligned} \quad (7.44)$$

We plug in the expressions (7.18), (7.40) for  $e_a^i$  and  $\bar{e}_a^i$  and apply the relations (7.5). For  $a > 1$ , (7.44) implies

$$k^3 q (\bar{\gamma}_a \gamma_{a+1} - \bar{\gamma}_{a+1} \gamma_a) r_{a-1} r_{a+1} r_a^{-2} \omega_{a-1}^{-1} \omega_{a+1}^{-1} = 0 \quad (7.45)$$

This means that in order for (7.41) to hold, we have to require

$$\bar{\gamma}_a \gamma_a^{-1} = \bar{\gamma}_{a+1} \gamma_{a+1}^{-1} \quad (7.46)$$

for every  $a > 1$ . A similar reasoning can be repeated for  $a \leq 1$  to show that (7.46) has to hold for any value of  $a$ . Therefore (7.43) is necessary.

When  $N$  is even, it is not possible to satisfy (7.41). This is a consequence of the fact that (7.42) does not hold for  $|a| = 1$  in this case, due to the particular form of  $\lambda_{\pm 1}$  and  $\bar{\lambda}_{\pm 1}$ . Choose the indices in (7.41) to be e.g.  $i = a = 1, j = b = 2$ . Plug in (7.18), (7.40) for  $e_a^i$  and  $\bar{e}_a^i$  and the definitions (5.14), (5.18) for  $\lambda_a$  and  $\bar{\lambda}_a$ , then apply (3.35) and (3.16). In this way (7.44) becomes

$$\begin{aligned} & -k^2 (\bar{\gamma}_2 \gamma_1 r_2^{-1} r_1^{-1} x^{-2} x^2 K^{-1} + k \bar{\gamma}_1 \gamma_2 q \omega_2^{-1} r_2 r_1^{-1} K) = \\ & -k^2 \bar{\gamma}_2 \gamma_1 r_2^{-1} r_1^{-1} (x^2 x^{-2} - k \omega_1^{-1} r_1^2) K^{-1} \end{aligned}$$

Due to the commutation relation (3.16) between  $x^2$  and  $x^{-2}$  the terms which are proportional to  $K^{-1}$  cancel and the following equation should be satisfied

$$k^3 q \bar{\gamma}_1 \gamma_2 \omega_2^{-1} r_2 r_1^{-1} K = 0.$$

But this would mean that either  $\bar{\gamma}_1$  or  $\gamma_2$  should vanish, which is not possible, if we want to have  $N$  independent objects  $\lambda_a$  instead of fewer. That is the reason why the theorem does not hold for  $N$  even.

In the case  $N$  odd (7.43) is consistent with (5.15), (5.19). We can determine  $c$  e.g. by applying (7.43) to  $a = 0$  and by recalling (5.15)<sub>1</sub>, (5.19)<sub>1</sub>. We thus find  $c = -q$ . In this way we get the last of the equations (5.23), which completely fixes the coefficients  $\bar{\gamma}_a$  in terms of the  $\gamma_a$ .

As the last step, we require (5.3). This imposes another condition relating  $\bar{\gamma}_a$  to  $\gamma_a$ :

$$\bar{\gamma}_a \gamma_a = k^{-2} \omega_{a-1} \omega_a q^{-1} \text{ for } a > 1, \quad (7.47)$$

$$\bar{\gamma}_1 \gamma_1 = h^{-2} q^{-1} \quad (7.48)$$

$$\bar{\gamma}_0 \gamma_0 = -h^{-2} \quad (7.49)$$

$$\bar{\gamma}_{-1} \gamma_{-1} = h^{-2} q \quad (7.50)$$

$$\bar{\gamma}_a \gamma_a = k^{-2} \omega_{a-1} \omega_a q \text{ for } a < -1, \quad (7.51)$$

Here we have used the expression (7.18) for  $e_a^a = \Lambda\varphi(\mathcal{L}^-_a)$  and (7.40) for  $\bar{e}_a^a = \Lambda\varphi(\mathcal{L}^+_a)$ . Equations (7.47) to (7.51) are compatible with (5.15), (5.19). If we replace  $\bar{\gamma}_a = -q\gamma_a$  we find the remaining equations in (5.23).

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